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EXAMPLES FOR PRACTICE IN THE USE OF
SEVEN-FIGURE LOGARITHMS.



EXAMPLES FOR PRACTICE

IN THE USE OF

SEVEN-FIGURE

LOGARITHMS

For the Use of Schools and Colleges

BY

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PREFATORY NOTE.

CONSIDERABLE practice is required by most students before they acquire the accuracy which is absolutely necessary in the use of logarithms. According to my experience, mistakes are more frequently made by taking out a wrong logarithm than by blundering in the arithmetic. It would seem therefore that a student should always be expected to take out himself all the logarithms, whether of numbers or of Trigonometrical Functions, which are needed in the solution of any question; and that the practice, which has been so common, of supplying him with all these in addition to the data should be abandoned. It is mainly with the wish of bringing about this result that I have put together the accompanying collection of examples. As the answers are given in each case, this little book is the result of a very considerable amount of work. I have taken all pains to secure accuracy, but cannot hope to have completely succeeded, and shall be grateful to anyone who will point out to me any mistakes I may have made.

I have given what may be thought a disproportionate number of examples of the calculation of the parts of a tetrahedron when

the lengths of its edges are given. This was done, chiefly because I am convinced that no other kind of examples so soon teaches the habitual accuracy which ought to be attained, but also to some extent to give examples for the use of those who have occasion for practice in the solution of Spherical Triangles. In the calculation of each tetrahedron is involved the obtaining all the parts of four Spherical Triangles, and this has induced me, after considerable hesitation, to abstain from giving explicit examples for such solutions.

I have principally, but not exclusively, used Chambers' Mathematical Tables in my working out of these examples. Of course the seventh significant place in numbers, and the second decimal place in seconds of angles, are not absolutely to be relied upon. If this book should be used with tables of six or five-figure logarithms, the corresponding number of places should be struck off my results, the usual correction being made when necessary in the remaining final figure.

COOPER'S HILL COLLEGE, *Sept. 4th*, 1888.

EXAMPLES FOR PRACTICE IN THE USE OF SEVEN-FIGURE LOGARITHMS

I.—Calculation of $L \sin$, $L \tan$, etc., when the angle is given, and conversely.

If it is required to find the $L \sin$, $L \cos$, $L \tan$, ..., of an angle not exactly given in the Tables, it is best to take out from the Tables the $L \sin$, $L \cos$, or $L \tan$ of that angle in the Tables which is next below the proposed angle; and then to calculate the "proportional part" for the seconds by a small sum in Practice worked at the side: this proportional part being then added (with the proper sign) to the logarithm taken out. Thus to calculate $L \tan 28^\circ 40' 49.2''$, $L \sin 27^\circ 45' 43.54''$, $L \cos 54^\circ 57' 58.2''$.

$\begin{array}{r} L \tan 28^\circ 40' = 9.7377714 \\ \text{diff. for } 49.2'' = \quad 2461 \\ \hline L \tan 28^\circ 40' 49.2'' = 9.7380175 \end{array}$	$\left \begin{array}{l} 60'' = 3001 \\ 48'' = \frac{48}{100} = 2400 \\ 1.2'' = \frac{1.2}{100} = 60 \end{array} \right \begin{array}{l} \\ 8 \\ 60 \end{array}$	$\begin{array}{r} L \sin 27^\circ 45' = 9.6680265 \\ \text{diff. for } 43 \text{ } 54'' = \quad 1742 \\ \hline L \sin 27^\circ 45' 23 \text{ } 54'' = 9.6682007 \end{array}$
$\begin{array}{r} 42'' = \frac{42}{100} = 1680 \\ 1.5'' = \frac{1.5}{100} = 60 \\ .04'' = \frac{.04}{100} = 1 \\ \hline 1742 \end{array}$	$\left \begin{array}{l} 60'' = 2400 \\ 42'' = \frac{42}{100} = 1680 \\ 1.5'' = \frac{1.5}{100} = 60 \\ .04'' = \frac{.04}{100} = 1 \end{array} \right \begin{array}{l} \\ 6 \\ 6 \\ 1 \end{array}$	$\begin{array}{r} L \cos 54^\circ 57' = 9.7591321 \\ \text{diff. for } 58.2'' = \quad 1746 \\ \hline 9.7589575 \end{array}$
		$\begin{array}{r} 60'' = 1800 \\ 1.8'' = \frac{1.8}{100} = 54 \end{array}$

Each of these examples might also be done by multiplying in each case the number of seconds by the quotient obtained by dividing by 60 the difference for 60". Thus, in the last example, this quotient is 30 and $58.2 \times 30 = 1746$. When the difference for 60" contains only three figures, this method is generally the shorter one. When the $L \sin$, $L \cos$, $L \tan$, ..., of an angle is given, and the angle is required, look out in the Tables the $L \sin$ (or $L \cos$ or $L \tan$, ...) *nearest* to the given one, whether greater or less, and write this under the given one, *omitting* those figures which are the same in both. Then take out the corresponding angle (putting + or - after it according as the number of seconds due to the difference is to be added or subtracted), and

the difference for 60"; and calculate the number of seconds by the usual proportion. Thus having given—

$$(1) \quad L \sin x = 9.8775429 \mid 1100 \mid 10320(9.38", \text{ or } x = 48^\circ 57' 50.62";$$

$$\begin{array}{r|l} 48^\circ 58' - \} & \frac{601}{172} \\ 1100 \} & \frac{42}{90} \end{array}$$

$$(2) \quad L \cos x = 9.7076837 \mid 2130 \mid 46380(21.78, \text{ or } x = 59^\circ 19' 38.22";$$

$$\begin{array}{r|l} 59^\circ 20' - \} & \frac{064}{773} \\ 2130 \} & \frac{378}{165} \\ & 159 \end{array}$$

$$(3) \quad L \sec x = 10.1324081 \mid 1158 \mid 25400(18.48, \text{ or } x = 42^\circ 30' 18.48";$$

$$\begin{array}{r|l} 42^\circ 30' + \} & \frac{3691}{390} \\ 1158 \} & \frac{9820}{556} \\ & 928 \end{array}$$

The method of "proportional parts" must not be employed for the $L \sin$ or $L \tan$ of an angle less than $3^\circ 30'$, the methods for "small" angles being then employed; as also for the $L \cos$ or $L \sec$ of an angle greater than $86^\circ 30'$.

[Not many examples for working are given under this heading, as every example in all the subsequent sections will furnish one or more.]

EXAMPLES. I.

1. Find the $L \sin$ of the following angles :—

$$68^\circ 47' 41.5", \quad 73^\circ 21' 23.91", \quad 78^\circ 29' 43.85", \quad 139^\circ 21' 10.74".$$

$$[\text{Ans. } 9.9695517, \quad 9.9814136, \quad 9.9911858, \quad 9.8138458.]$$

2. Find the $L \cos$ of the following angles :—

$$25^\circ 45' 33.61", \quad 22^\circ 24' 19.04", \quad 18^\circ 49' 46.32", \quad 66^\circ 59' 38.97.$$

$$[\text{Ans. } 9.9545452, \quad 9.965912, \quad 9.9761128, \quad 9.59919823.]$$

3. Find the $L \tan$ of the following angles :—

$$23^\circ 48' 41.12", \quad 46^\circ 0' 43.79", \quad 66^\circ 11' 18.88", \quad 43^\circ 59' 16.21".$$

$$[\text{Ans. } 9.6447248, \quad 10.0153473, \quad 10.3552752, \quad 9.9846527.]$$

4. Find the value of x from each of the following equations :—

$$(a) \quad L \sin x = 9.5740674, \quad L \sin x = 9.8091259, \quad L \sin x = 9.8281920, \quad L \sin x = 9.9860122;$$

$$(b) \quad L \tan x = 9.3600730, \quad L \tan x = 9.8353762, \quad L \tan x = 10.0562416, \quad L \tan x = 10.1321843;$$

$$(c) \quad L \cos x = 9.9670225, \quad L \cos x = 9.8610172, \quad L \cos x = 9.7544211, \quad L \cos x = 9.3701332.$$

$$[\text{Ans. } \begin{cases} (a) \quad 22^\circ 1' 34.47", \quad 40^\circ 7' 2.6", \quad 42^\circ 19' 13.01", \quad 75^\circ 32' 9.83"; \\ (b) \quad 12^\circ 54' 18.65", \quad 34^\circ 23' 30.51", \quad 48^\circ 41' 58.59", \quad 53^\circ 35' 16.23"; \\ (c) \quad 21^\circ 51' 1.23", \quad 43^\circ 26' 12.02", \quad 59^\circ 22' 56.98", \quad 76^\circ 26' 17.35". \end{cases}]$$

5. Find the values of the following fractions :—

$$(1) \quad \frac{\sin 58^\circ 9' 20.9'' \sin 49^\circ 26' 47.92'' \sin 65^\circ 18' 24.84''}{\sin 57^\circ 11' 25.7'' \sin 53^\circ 17' 18.06'' \sin 60^\circ 29' 59.54''}$$

$$(2) \quad \frac{\sin 57^\circ 11' 25.7'' \sin 61^\circ 20' 39.56'' \sin 68^\circ 33' 21.04''}{\sin 58^\circ 9' 20.9'' \sin 57^\circ 30' 9.46'' \sin 73^\circ 21' 46.38''}$$

$$(3) \quad \frac{\sin 61^\circ 24' 17.1'' \sin 57^\circ 30' 9.46'' \sin 60^\circ 29' 59.54''}{\sin 53^\circ 56' 29.5'' \sin 61^\circ 20' 39.56'' \sin 65^\circ 18' 24.84''}$$

$$(4) \quad \frac{\sin 53^\circ 56' 29.5'' \sin 53^\circ 17' 18.06'' \sin 73^\circ 21' 46.38''}{\sin 61^\circ 24' 17.1'' \sin 49^\circ 26' 47.92'' \sin 68^\circ 33' 21.04''}$$

[Each fraction=1.]

6. Prove the truth of the equations :—

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} = 2 \sqrt{\frac{\sin s \sin (s-a) \sin (s-b) \sin (s-c)}{\sin a \sin b \sin c}}, \quad (2s = a + b + c),$$

for the following systems :—

$$\begin{aligned} (1) \quad & \left\{ \begin{array}{l} a = 27^{\circ} 28' 41.02'', \quad A = 49^{\circ} 24' 1.48'' \\ b = 33^{\circ} 18' 34.24'', \quad B = 64^{\circ} 38' 45.10'' \\ c = 35^{\circ} 42' 29.32'', \quad C = 73^{\circ} 49' 49.80'' \end{array} \right\}; \quad (2) \quad \left\{ \begin{array}{l} a = 50^{\circ} 0' 30.76'', \quad A = 49^{\circ} 24' 1.48'' \\ b = 62^{\circ} 59' 51.52'', \quad B = 62^{\circ} 0' 25.14'' \\ c = 81^{\circ} 9' 58.50'', \quad C = 101^{\circ} 40' 57.08'' \end{array} \right\}; \\ (3) \quad & \left\{ \begin{array}{l} a = 81^{\circ} 17' 39.16'', \quad A = 77^{\circ} 15' 8.76'' \\ b = 66^{\circ} 19' 31.38'', \quad B = 64^{\circ} 38' 45.10'' \\ c = 97^{\circ} 2' 17.90'', \quad C = 101^{\circ} 40' 57.08'' \end{array} \right\}; \quad (4) \quad \left\{ \begin{array}{l} a = 65^{\circ} 31' 27.26'', \quad A = 77^{\circ} 15' 8.76'' \\ b = 55^{\circ} 29' 1.08'', \quad B = 62^{\circ} 0' 25.14'' \\ c = 63^{\circ} 39' 57.86'', \quad C = 73^{\circ} 49' 49.80'' \end{array} \right\}; \end{aligned}$$

also prove that the value of each member of the equations is (1) 1.64556, (2) .991042, (3) .9867222, (4) 1.071654.

II.—Calculation of $\sqrt{a^2 + b^2}$, when a, b are given numbers.

Taking u to denote $\sqrt{a^2 + b^2}$, then if θ be an angle determined by the equation $\tan \theta = \frac{b}{a}$, $u = a \sec \theta = b \operatorname{cosec} \theta$. The value of u may be determined from either of these equations; if both be used, the calculation is thereby checked. The work may be conveniently arranged as in the following example.

$a = 47993.27$ $b = 38932.89$	$\log b = 4.5903167$ $\log a = 4.6811804$ <hr style="border: none; border-top: 1px solid black;"/> $L \tan \theta = 9.9091363$ $39^{\circ} 3' - \}$ $2582 \quad \}$ <div style="text-align: right; padding-right: 10px;">440 77</div>	$4620(1.79$ 2038 <hr style="border: none; border-top: 1px solid black;"/> 2306 $\theta = 39^{\circ} 2' 58.21''$	$\log \sec \theta = .1098046$ <div style="text-align: right; padding-right: 10px;">31</div> <hr style="border: none; border-top: 1px solid black;"/> $\log a = 4.6811804$ <hr style="border: none; border-top: 1px solid black;"/> $\log u = 4.7909819$	$1025 = 60''$ $1.8 = 1^{\circ} 48' = 30$ <div style="text-align: right; padding-right: 10px;">8</div> <hr style="border: none; border-top: 1px solid black;"/> $31 \quad \}$ <div style="text-align: right; padding-right: 10px;">2</div>
	$\log \operatorname{cosec} \theta = .2006606$ <div style="text-align: right; padding-right: 10px;">46</div> <hr style="border: none; border-top: 1px solid black;"/> $\log b = 4.5903167$ <hr style="border: none; border-top: 1px solid black;"/> $\log u = 4.7909819$	$1558 = 60''$ <div style="text-align: right; padding-right: 10px;">7</div> <hr style="border: none; border-top: 1px solid black;"/> $1.8 = 46$ <div style="text-align: right; padding-right: 10px;">3</div> <hr style="border: none; border-top: 1px solid black;"/> $46 \quad \}$		

The two calculations agree in giving $\log u = 4.7909819$, which is therefore almost certainly correct. This gives $u = 61799.07$.

EXAMPLES. II.

DATA.	ANSWERS.
1. $a = 571.4284, \quad b = 428.5713.$	714.2854 ($\theta = 36^{\circ} 52' 11.65''$).
2. $a = 93921.58, \quad b = 37917.43.$	101286.7 ($\theta = 21^{\circ} 59' 4.87''$).
3. $a = 4285.714, \quad b = 5714.285.$	7142.855 ($\theta = 36^{\circ} 52' 11.65''$).
4. $a = 23727.92, \quad b = 7873.11.$	25000.01 ($\theta = 18^{\circ} 21' 22.5''$).
5. $a = 7139.247, \quad b = 3143.892.$	7800.827 ($\theta = 23^{\circ} 46' 1.68''$).
6. $a = 52791.39, \quad b = 37387.41.$	64689.64 ($\theta = 35^{\circ} 18' 23.46''$).
7. $a = 171428.4, \quad b = 169999.8.$	241428.3 ($\theta = 44^{\circ} 45' 36.92''$).
8. $a = 3081.708, \quad b = 212.7818.$	3089.046 ($\theta = 3^{\circ} 56' 59.34''$).
9. $a = 9853.08, \quad b = 2127.818.$	10080.22 ($\theta = 12^{\circ} 11' 10.14''$).
10. $a = 13790.4, \quad b = 13790.3.$	19502.5 ($\theta = 44^{\circ} 59' 59.26''$).
11. $a = 803.761, \quad b = 803.76.$	1136.689 ($\theta = 44^{\circ} 59' 59.88''$).

EXAMPLES. II.—Continued.

DATA.	ANSWERS.
12. $a=7118.12$, $b=606.13$.	7143.878 ($\theta=4^{\circ}52'1.82''$).
13. $a=34285.68$, $b=9999.99$.	35714.25 ($\theta=16^{\circ}15'36.75''$).
14. $a=7372.46$, $b=6989.37$.	10158.95 ($\theta=43^{\circ}28'19.35''$).
15. $a=1185.184$, $b=345.6796$.	1234.567 ($\theta=16^{\circ}15'36.88''$).
16. $a=273.0004$, $b=135.9992$.	305. ($\theta=26^{\circ}28'51.17''$).
17. $a=272.9$, $b=130.2022$.	305.0008 ($\theta=26^{\circ}31'24.3''$).
18. $a=5488.42$, $b=2273.378$.	5940.623 ($\theta=22^{\circ}30'$).
19. $a=378.4125$, $b=487.4775$.	615.37 ($\theta=52^{\circ}5'4.83''$).
20. $a=27891.83$, $b=36789.17$.	46167.06 ($\theta=52^{\circ}49'56.27''$).
21. $a=262.087$, $b=151.316$.	302.632 ($\theta=29^{\circ}59'59.98''$).
22. $a=56471.9$, $b=97812.2$.	112943.8 ($\theta=60''$).
23. $a=8055.837$, $b=5924.819$.	10000 ($\theta=36^{\circ}20'$).
24. $a=7601.54$, $b=6497.43$.	10000 ($\theta=40^{\circ}31'20''$).
25. $a=7968.81$, $b=23695.95$.	25000 ($\theta=71^{\circ}24'45.02''$).
26. $a=1236.113$, $b=716.1294$.	1428.572 ($\theta=30^{\circ}5'7.5''$).
27. $a=74511.63$, $b=19109.62$.	76923.095 ($\theta=14^{\circ}23'3.6''$).
28. $a=4313.546$, $b=3999.424$.	5882.351 ($\theta=42^{\circ}50'9.6''$).
29. $a=4127.817$, $b=1365.548$.	4347.825 ($\theta=18^{\circ}18'18.17''$).
30. $a=437.7493$, $b=796.7571$.	909.0909 ($\theta=61^{\circ}12'54''$).
31. $a=6151.67$, $b=1104.292$.	6250.001 ($\theta=10^{\circ}10'36.6''$).
32. $a=4761.903$, $b=13468.55$.	14285.67 ($\theta=70^{\circ}31'42.88''$).

III.—Solution of Equations of the form $a \cos x + b \sin x = c$.

The two values of x which satisfy the equation $a \cos x + b \sin x = c$ being denoted by α , β , $\frac{1}{a} \cos \frac{1}{2}(\alpha + \beta) = \frac{1}{b} \sin \frac{1}{2}(\alpha + \beta) = \frac{1}{c} \cos \frac{1}{2}(\alpha - \beta)$. First find $\frac{1}{2}(\alpha + \beta)$ from the equation $\tan \frac{1}{2}(\alpha + \beta) = \frac{b}{a}$, and then $\frac{1}{2}(\alpha - \beta)$ from either or both of the equations $\cos \frac{1}{2}(\alpha - \beta) = \frac{c}{a} \cos \frac{1}{2}(\alpha + \beta) = \frac{c}{b} \sin \frac{1}{2}(\alpha + \beta)$. Writing the equation so that c is positive, then if a be negative and b positive, put $x = \pi - y$; if a be positive and b negative, put $x = -y$; and if both a and b be negative, put $x = \pi + y$. We shall thus always have an equation in which the coefficients are all positive. Arrange the work according to the following pattern.

$a=27839$ $b=29273$ $c=28565$	$L \tan \alpha + \beta = \frac{14.4664672 \dots (10 + \log b)}{-4.4446536 \dots (\log a)}$ $\frac{7380}{7380}$ $\frac{756 \times 6\phi}{4536(17.93)}$ $\frac{2006}{235}$ 73	$\frac{9.8383441}{-397}$ $\frac{4.4558342}{-4.4446536}$ $\frac{9.8494850}{=L \cos \frac{1}{2}(\alpha - \beta)}$ $=L \cos 45^{\circ}$	$\frac{1329-60''}{18=398}$ $\frac{7}{-0.067-1}$ $\frac{3}{-.01=-}$ $\frac{2}{397}$
		$\frac{9.8600821}{359}$ $\frac{4.4558342}{-4.4664672}$ $\frac{9.8494850}{=L \cos \frac{1}{2}(\alpha - \beta)}$	$\frac{1201=60''=20\frac{1}{6} \times 60}{358.6}$ $\frac{3}{359}$

Thus $\frac{a+\beta}{2} = 46^{\circ}26'17.93''$, $\frac{a-\beta}{2} = 45^{\circ}$, } whence $a = 91^{\circ}26'17.93''$, $\beta = 1^{\circ}26'17.93''$.

If the values of a, b, c be such that $c^2 > a^2 + b^2$, the values of x will be impossible. Should such data be given, the value found for $I \cos \frac{1}{2}(a - \beta)$ would be greater than 10, and the corresponding angle impossible.

EXAMPLES. III.

DATA.	ANSWERS.
1. $a = 11175.3$, $b = 5600.24$, $c = 10548.25$.	$a = 59^{\circ}4'$, $\beta = 5^{\circ}50'$.
2. $a = 58533.32$, $b = 49909.95$, $c = 62137.14$.	$a = 76^{\circ}34'24.7''$, $\beta = 4^{\circ}20'$.
3. $a = 67313.97$, $b = 58129.13$, $c = 62721.45$.	$a = 85^{\circ}57'38.3''$, $\beta = -4^{\circ}20'9.6''$.
4. $a = 39713.76$, $b = 29254.24$, $c = 18397.98$.	$a = 104^{\circ}28'34.01''$, $\beta = -31^{\circ}43'24.07''$.
5. $a = 6342.667$, $b = 3914.217$, $c = 6549.328$.	$a = 60^{\circ}11'27.25''$, $\beta = 3^{\circ}10'7.07''$.
6. $a = 549.1238$, $b = 508.1741$, $c = 618.3272$.	$a = 77^{\circ}2'50.77''$, $\beta = 8^{\circ}30'59.71''$.
7. $a = 35.67812$, $b = 28.73407$, $c = 40.13598$.	$a = 67^{\circ}40'1.8''$, $\beta = 10^{\circ}1'36''$.
8. $a = 7891.293$, $b = 889.312$, $c = 7926.027$.	$a = 9^{\circ}58'39.75''$, $\beta = 2^{\circ}52'55.13''$.
9. $a = 3712.484$, $b = 1139.115$, $c = 3795.726$.	$a = 29^{\circ}14'59.21''$, $\beta = 4^{\circ}51'56.79''$.
10. $a = 801.3982$, $b = 116.4593$, $c = 869.154$.	$a = 8^{\circ}28'58.52''$, $\beta = 6^{\circ}54'58.52''$.
11. $a = 25130.85$, $b = 23846.16$, $c = 27389.27$.	$a = 81^{\circ}15'31.62''$, $\beta = 6^{\circ}23'20.84''$.
12. $a = 1790.009$, $b = 2811.935$, $c = 2956.161$.	$a = 85^{\circ}2'25.2''$, $\beta = 30''$.
13. $a = 78913.47$, $b = 59134.72$, $c = 67314.95$.	$a = 83^{\circ}47'49.92''$, $\beta = -10^{\circ}6'15.32''$.
14. $a = 3581.72$, $b = 3312.93$, $c = 4645.07$.	$a = 60^{\circ}34'48.14''$, $\beta = 24^{\circ}57'17.40''$.
15. $a = 428.5714$, $b = 571.4286$, $c = 714.2857$.	$a = 53^{\circ}48.35''$, $\beta = 53^{\circ}7'48.35''$.
16. $a = 20785.2$, $b = 20611.99$, $c = 20058.41$.	$a = 89^{\circ}2'13.18''$, $\beta = 0^{\circ}29'0.70''$.
17. $a = 9999.79$, $b = 8654.33$, $c = 11235.84$.	$a = 72^{\circ}42'19.50''$, $\beta = 9^{\circ}2'37.46''$.
18. $a = 37912.37$, $b = 28776.52$, $c = 39148.21$.	$a = 71^{\circ}51'49.02''$, $\beta = 2^{\circ}32'7.22''$.
19. $a = 84513.27$, $b = 9837.12$, $c = 79125.64$.	$a = 28^{\circ}12'10.98''$, $\beta = -14^{\circ}55'28.70''$.
20. $a = 831.2798$, $b = 415.6399$, $c = 888.8931$.	$a = 44^{\circ}25'31.66''$, $\beta = 8^{\circ}22'32.54''$.
21. $a = 813908.5$, $b = 600305.1$, $c = 1000000$.	$a = 45^{\circ}0'0.04''$, $\beta = 27^{\circ}49'18.4''$.
22. $a = 7933.533$, $b = 6087.614$, $c = 9914.449$.	$a = 45''$, $\beta = 30''$.
23. $a = 7261.248$, $b = 6875.629$, $c = 9015.867$.	$a = 50^{\circ}52'30''$, $\beta = 36^{\circ}0'0.0''$.
24. $a = 35170.77$, $b = 35538.95$, $c = 48363.02$.	$a = 60''$, $\beta = 30^{\circ}35'48.0''$.
25. $a = 786.1347$, $b = 563.9187$, $c = 803.4125$.	$a = 60^{\circ}31'40.06''$, $\beta = 1^{\circ}46'41.84''$.
26. $a = 43759.16$, $b = 24189.58$, $c = 44576.71$.	$a = 92^{\circ}0'0.00''$, $\beta = -34^{\circ}8'0.00''$.
27. $a = 137904$, $b = 137903$, $c = 195025$.	$a = 44^{\circ}59'59.24''$, $\beta = 44^{\circ}59'59.24''$.
28. $a = 803.76$, $b = 803.761$, $c = 1136.689$.	$a = 45^{\circ}0'0.12''$, $\beta = 45^{\circ}0'0.12''$.
29. $a = 70$, $b = 24$, $c = 37$.	$a = 78^{\circ}55'28.72''$, $\beta = -41^{\circ}4'31.28''$.
30. $a = 70$, $b = 24$, $c = 64.08587$.	$a = 48^{\circ}55'28.72''$, $\beta = -11^{\circ}4'31.28''$.
31. $a = 39405.98$, $b = 17622.89$, $c = 21583.55$.	$a = 21^{\circ}54'18.56''$, $\beta = 95^{\circ}54'18.6''$.
32. $a = 97.8122$, $b = 56.4719$, $c = 112.9438$.	$a = 30''$, $\beta = 30''$.
33. $a = 27442$, $b = 11366.89$, $c = 29703.12$.	$a = 22^{\circ}30'$, $\beta = 22^{\circ}30'$.
34. $a = 5488.43$, $b = 273.382$, $c = 5940.635$.	$a = 22^{\circ}30'$, $\beta = 22^{\circ}30'$.
35. $a = 43155.16$, $b = 68971.5$, $c = 71859.22$.	$a = 85^{\circ}55'55''$, $\beta = 30''$.
36. $a = 395.76$, $b = 228.48$, $c = 323.12$.	$a = 75^{\circ}0'3.18''$, $\beta = -15^{\circ}0'12.68''$.
37. $a = 193.19$, $b = 198.79$, $c = 196.01$.	$a = 90^{\circ}49'6.6''$, $\beta = -0^{\circ}49'6.6''$.
38. $a = 27.839$, $b = 28.273$, $c = 28.565$.	$a = 89^{\circ}23'44.83''$, $\beta = 1^{\circ}29'25.83''$.
39. $a = 239.9$, $b = 3992.1$, $c = 3568.9$.	$a = 113^{\circ}23'13.59''$, $\beta = 59^{\circ}44'5.71''$.
40. $a = 11.599$, $b = 49.121$, $c = 35.689$.	$a = 121^{\circ}42'50.56''$, $\beta = 31^{\circ}42'50.56''$.
41. $a = 3496.7$, $b = 63497$, $c = 5125.7$.	$a = 106^{\circ}41'30.68''$, $\beta = -16^{\circ}0'18.28''$.
42. $a = 34967$, $b = 63497$, $c = 51257$.	$a = 106^{\circ}9'32.31''$, $\beta = 16^{\circ}9'32.31''$.

IV.—Calculation of $\sqrt{b^2+c^2-2bc \cos A}$.

Denoting the expression by u , its value may be calculated by any one of the three following methods :—

$$(1) \quad u_1 = (b+c) \cos \theta, \text{ where } \sin \theta = \frac{2\sqrt{bc} \cos \frac{1}{2}A}{b+c};$$

$$(2) \quad u_2 = (b-c) \sec \theta, \text{ where } \tan \theta = \frac{2\sqrt{bc}}{b-c} \sin \frac{1}{2}A; \quad (b > c)$$

$$(3) \quad u_3 = (b+c) \sin \frac{1}{2}A \sec \theta, \text{ where } \tan \theta = \frac{b-c}{b+c} \cot \frac{1}{2}A;$$

The calculation of one example by each of these three methods is subjoined.

DATA.

$$b=12369.12$$

$$c=4123.04$$

$$A=40^\circ 55' 55''$$

$$b+c=16492.16$$

$$b-c=8246.08$$

$$A=20^\circ 27' 57.5''$$

$$2$$

$$=20^\circ 28' 2.5''$$

$$\log b=4.0923388$$

$$\log c=3.6152175$$

$$2)7.7075563$$

$$\log \sqrt{bc}=3.8537782$$

$$\log (b+c)=4.2172775$$

$$\log (b-c)=3.9162475$$

(1)

$$.30103$$

$$3.8537782$$

$$9.9716820$$

$$19)$$

$$-4.2172775$$

$$471$$

$$2.4-18 \quad 8$$

$$.1- \quad 7$$

$$2.5-19 \quad 5$$

$$\log \cos \theta=1.7667739$$

$$1753$$

$$432$$

$$\log (b+c)=4.2172775$$

$$10-292 \quad 2$$

$$4.8-140 \quad 2$$

$$\log u_1=3.9840946$$

$$-.03=1 \quad 2$$

$$05$$

$$14.84=4.32$$

$$L \sin \theta=9.9092146$$

$$)1350(14.84''$$

$$54^\circ 14' - \} 371$$

$$910$$

$$440$$

$$76$$

$$32$$

$$9640.3$$

$$9- \quad 41$$

$$u_1=9640.390$$

(2)

$$.3010300$$

$$3.8537782$$

$$9.5436489$$

$$-141)$$

$$-5.9162475$$

$$3386$$

$$2 -113$$

$$.5- 28$$

$$2.5=141$$

$$\log \sec \theta=.0678489$$

$$765$$

$$-16$$

$$\log (b-c)=3.9162475$$

$$1.2=15 \quad 3$$

$$.02= \quad 2$$

$$\log u_2=3.9840948$$

$$15 \quad 5$$

$$L \tan \theta=9.7821955$$

$$)3480(1.22$$

$$31^\circ 12' - \}$$

$$2851$$

$$2013$$

$$58$$

$$629$$

$$588$$

$$u_2=9640.394$$

(3)

$$3.9162475$$

$$10.4280331$$

$$161)$$

$$-4.2172775$$

$$3858$$

$$2 -128 \quad 6$$

$$.5= 32 \quad 2$$

$$2.5=161$$

$$\log \sec \theta=.2232324$$

$$1693$$

$$-498$$

$$\log \sin \frac{1}{2}A=1.5436348$$

$$15 =423 \quad 3$$

$$\log (b+c)=4.2172775$$

$$2 = 56 \quad 4$$

$$.66= 18 \quad 6$$

$$\log u_3=3.9840949$$

$$17.66=498$$

$$L \tan \theta=10.1270192$$

$$)46560(17.66$$

$$53^\circ 16' - \}$$

$$2636$$

$$968$$

$$776$$

$$2020$$

$$1748$$

$$1664$$

$$u_3=9640.396$$

The three values of $\log u$, found according to the three methods, will generally have slight differences in the last place. In the examples given, the answers will exhibit the three values of u , denoted by u_1 , u_2 , u_3 ; and the mean of these is taken for the value of u . Thus, in the one calculated, $u_1 = 9640.390$, $u_2 = 9640.394$, $u_3 = 9640.396$, $u = 9640.393$.

EXAMPLES. IV.

DATA.	ANSWERS.
1. $b=45123.76$, $c=34179.24$, $A=24^\circ 57' 18.6''$.	$u_1=20193.17$, $u_2=u_3=20193.20$, $u=20193.18$.
2. $b=2357.126$, $c=2075.352$, $A=67^\circ 31' 55''$.	$u_1=2474.689$, $u_2=u_3=2474.691$, $u=2474.69$.
3. $b=2357.126$, $c=2075.382$, $A=67^\circ 31' 55''$.	$u_1-u_3=u_2-u=2474.705$.
4. $b=47392.16$, $c=25497.83$, $A=56^\circ 37' 55.2''$.	$u_1=39583.89$, $u_2=u_3=39853.92$, $u=39853.9$.
5. $b=38914.86$, $c=34819.73$, $A=120^\circ$.	$u_1=63888.85$, $u_2=63888.91$, $u_3=63888.89$, $u=63888.88$.
6. $b=73869.13$, $c=9347.25$, $A=60^\circ$.	$u_1=69667.42$, $u_2=u_3=69667.40$, $u=69667.41$.
7. $b=543.2971$, $c=276.8139$, $A=36^\circ 55' 57.6''$.	$u_1=362.4475$, $u_2=362.4477$, $u_3=362.447$, $u=362.4474$.
8. $b=47394.91$, $c=21382.75$, $A=26^\circ 39' 45''$.	$u_1=29869.01$, $u_2=u_3=29869.03$, $u=29869.02$.
9. $b=7984.637$, $c=4138.363$, $A=31^\circ 37' 52.8''$.	$u_1=4961.009$, $u_2=u_3=4961.012$, $u=4961.011$.
10. $b=73182.2$, $c=47297.4$, $A=39^\circ 19' 5.62''$.	$u_1=47297.42$, $u_2=u_3=47297.39$, $u=47297.4$.
11. $b=670.344$, $c=580.551$, $A=120^\circ$.	$u_1-u_3=u_2-u=1080.309$.
12. $b=3791.298$, $c=413.115$, $A=4^\circ 30'$.	$u_1=3379.611$, $u_2=3379.69$, $u_3=3379.612$, $u=3379.611$.
13. $b=3789.371$, $c=1134.199$, $A=23^\circ 6' 16.8''$.	$u_1=2781.982$, $u_2=u_3=2781.98$, $u=2781.981$.
14. $b=890.8505$, $c=534.1539$, $A=53^\circ 7' 48.38''$.	$u_1=718.6850$, $u_2=718.6852(=u)$, $u_3=718.6855$.

Other examples for practice may be taken from VII.

V.—Triangles in which are given two angles and one side.

Let a be the given length of one side; then whichever two of the three angles are given, the third is at once found from the equation $A+B+C=180^\circ$; then find R the radius of the circumscribed circle from the equation $2R = a/\sin A$, and the other sides b , c from the equations $b = 2R \sin B$, $c = 2R \sin C$. The radii of the inscribed and escribed circles r , r_1 , r_2 , r_3 , if required, may best be found from the equations $r = (s-a) \tan \frac{A}{2}$, $r_1 = s \tan \frac{A}{2}$, $r_2 = (s-c) \cot \frac{A}{2}$, $r_3 = (s-b) \cot \frac{A}{2}$, or the corresponding system involving $\tan \frac{B}{2}$ or $\tan \frac{C}{2}$, taking that one of the three half angles which is nearest to 45° . The equation $r_1 + r_2 + r_3 - r = 4R$ should then be employed to check the calculations. An example is worked out at full length below.

DATA.		
$a=27837.93$	$\log a = 4.4446369$	$\frac{277}{-}$
$A=77^\circ 36' 57.6''$	$-\log \sin A = 1.9897766$	$2.4 = 1\frac{4}{10} = 11 \quad 1$
$B=54^\circ 22' 12.48''$		
hence $C=48^\circ 0' 49.92''$	$\log 2R = 4.4548614$	$\frac{01}{-}$
	<u>28501.09.</u>	<u>13</u>

$\log 2R = 4.4548614$ $\log \sin B = 1.9099634$ <hr/> $\log b = 4.3648436$ <hr/> $23165.60.$ <hr/> 113	905 $12'' = 181$ $.48 = 7 \quad 2$	$\log 2R = 4.4548614$ $\log \sin C = 1.8710735$ <hr/> $\log c = 4.3260295$ <hr/> $21185.05.$ <hr/> 10	1137 $48 = 909 \quad 6$ $1.8 = 34 \quad 1$ $.12 = 2 \quad 3$
$a = 27837.93$ $b = 23165.60$ $c = 21185.05$ <hr/> $2s = 72188.58$ <hr/> $s = 36094.29$	$s - a = 8256.36$ $s - b = 12928.69$ $s - c = 14909.24$ <hr/> $s = 36094.29$	$\frac{A}{2} = 38^\circ 48' 28.8''$ $L \tan \frac{1}{2} A = 9.9052672$ <hr/> $\log \tan \frac{1}{2} A = 1.9053914$ <hr/> $\log \cot \frac{1}{2} A = 0.0946086$	$\log s - a = 3.9167886$ $\log s - b = 4.1115545$ $\log s - c = 4.1734555$ <hr/> $\log s = 4.5574385$
$\log s - a = 3.9167886$ $\log \tan \frac{1}{2} A = 1.9053914$ <hr/> $\log r = 3.8221800$ <hr/> $6640.182.$ <hr/> -12	$\log s = 4.5574385$ $\log \tan \frac{1}{2} A = 1.9053914$ <hr/> $\log r_1 = 4.4628299$ <hr/> $29028.85.$ <hr/> 128	$\log s - c = 4.1734555$ $\log \cot \frac{1}{2} A = 0.0946086$ <hr/> $\log r_2 = 4.2686041$ <hr/> $18538.05.$ <hr/> 12	$\log s - b = 4.1115545$ $\log \cot \frac{1}{2} A = 0.0946086$ <hr/> $\log r_3 = 4.2061631$ <hr/> $16075.45.$ <hr/> 121
ANSWERS. $C = 48^\circ 0' 49.92''$ $b = 23165.66.$ $c = 21185.05.$ $R = 14250.54.$		CHECK. $-r = -6640.18$ $r_1 = 29028.85$ $r_2 = 18538.05$ $r_3 = 16075.45$ <hr/> $4R = 57002.17$ <hr/> $R = 14250.54$	

The distance between the circumcentre, orthocentre, and incentre are to be found, when required, from the equations $\Delta_1^2 = R^2 - 2Rr$, $\Delta_2^2 = R^2 + 2\rho^2$, $\Delta_3^2 = \rho^2 + 2r^2$, where $\rho^2 = -4R^2 \cos A \cos B \cos C$.

EXAMPLES. V.

DATA.	ANSWERS.
1. $a = 7$ $R = 38^\circ 36'$ $C = 129^\circ 23'$	$A = 12^\circ 1'$ $b = 20.97616$ $c = 25.98715$ $R = 16.81107$ $r = 2.103095$ $r_1 = 2.839854$ $r_2 = 9.448844$ $r_3 = 57.05867$
2. $a = 86602.54$ $B = 80^\circ 42' 17.3''$ $C = 39^\circ 17' 42.7''$	$A = 60^\circ$ $b = 98686.93$ $c = 63331.6$ $R = 50000$ $r = 21770.725$ $r_1 = 71771.02$ $r_2 = 105618.62$ $r_3 = 44381.38$
3. $a = 91035.91$ $B = 37^\circ 43'$ $C = 66^\circ 43' 41.9''$	$A = 65^\circ 33' 18.1''$ $b = 61175.72$ $c = 91864.15$ $R = 50000$ $r = 20788.34$ $r_1 = 88339.88$ $r_2 = 41683.78$ $r_3 = 90764.7$

EXAMPLES. V.—Continued.

DATA.	ANSWERS.	
4. $a=96840.39$ $B=37^{\circ}43'$ $C=66^{\circ}43'28.3''$	$A=75^{\circ}33'31.7''$ $b=61175.71$ $c=91861.56$ $A'=50000$	$r=21779.2$ $r_1=96840.57$ $r_2=42674.64$ $r_3=82264$
5. $a=11935.26$ $B=41^{\circ}24'34.3''$ $C=24^{\circ}17'42.3''$	$A=114^{\circ}17'43.4''$ $b=8661.392$ $c=5387.97$ $A'=6547.5$	$r=1636.927$ $r_1=20119.62$ $r_2=4910.528$ $r_3=2796.707$
6. $a=22733.78$ $B=90^{\circ}$ $C=67^{\circ}30'$	$A=22^{\circ}30'$ $b=59406.23$ $c=54884.19$ $A'=29703.115$	$r=9105.87$ $r_1=13627.91$ $r_2=68512.1$ $r_3=45778.32$
7. $a=6347.287$ $A=78^{\circ}33'14.7''$ $B=39^{\circ}59'43.5''$	$C=61^{\circ}27'1.8''$ $b=4162.347$ $c=6404.116$ $A'=3238.04$	$r=1725.267$ $r_1=6916.216$ $r_2=2510.033$ $r_3=5251.178$
8. $a=12791.38$ $A=62^{\circ}37'15.5''$ $B=68^{\circ}25'37.24''$	$C=48^{\circ}57'7.26''$ $b=13395.9$ $c=10863.65$ $A'=7202.49$	$r=3487.813$ $r_1=11208.31$ $r_2=12596.28$ $r_3=8433.18$
9. $a=7674.893$ $A=97^{\circ}18'47.1''$ $C=40^{\circ}14'33.5''$	$B=42^{\circ}26'30.1''$ $b=5222.048$ $c=4998.842$ $A'=3868.919$	$r=1446.811$ $r_1=10169.62$ $r_2=3474.63$ $r_3=3278.239$
10. $a=44569.33$ $A=80^{\circ}32'57.54''$ $C=45^{\circ}36'2.36''$	$B=53^{\circ}51'0.1$ $b=30483.84$ $c=32282.09$ $A'=22591.29$	$r=10250.9$ $r_1=48014.57$ $r_2=28780.24$ $r_3=23821.26$
11. $a=8984.798$ $A=72^{\circ}48'48.6''$ $C=36^{\circ}36'12.72''$	$B=70^{\circ}34'58.68''$ $b=8869.832$ $c=5607.804$ $A'=4702.37$	$r=2025.334$ $r_1=8651.132$ $r_2=8303.540$ $r_3=3880.131$
12. $a=381.437$ $A=67^{\circ}10'25.8''$ $C=75^{\circ}31'14.5''$	$B=37^{\circ}18'19.7''$ $b=250.818$ $c=400.7024$ $A'=206.9235$	$r=89.677$ $r_1=342.9778$ $r_2=174.3436$ $r_3=400.0496$
13. $a=176.01$ $A=66^{\circ}19'31.38''$ $B=63^{\circ}39'57.86''$	$C=58^{\circ}0'30.76''$ $b=172.24$ $c=147.24$ $A'=96.0921$	$r=46.8754$ $r_1=161.8895$ $r_2=153.8063$ $r_3=115.5479$
14. $a=443.387$ $B=27^{\circ}28'41.02''$ $C=55^{\circ}29'1.08''$	$A=97^{\circ}2'17.90''$ $b=206.136$ $c=368.109$ $A'=223.3767$	$r=74.00344$ $r_1=575.4995$ $r_2=124.4027$ $r_3=267.6083$
15. $a=1266.82$ $A=81^{\circ}9'58.5''$ $C=33^{\circ}18'34.24''$	$B=65^{\circ}31'27.26''$ $b=1166.82$ $c=704.04$ $A'=641.0137$	$r=258.7082$ $r_1=1343.8572$ $r_2=1009.5812$ $r_3=469.3244$

EXAMPLES. V.—Continued.

DATA.	ANSWERS.
16. $\alpha=525.069$ $A=81^{\circ}17'39.16''$ $B=62^{\circ}59'59.52''$	$C=35^{\circ}42'29.32''$ $b=473.283$ $c=310.032$ $R=265.5946$ $r=110.8561$ $r_1=561.644$ $r_2=400.8712$ $r_3=210.7192$
17. $\alpha=122.073$ $A=64^{\circ}31'45.28''$ $B=65^{\circ}48'27.70''$	$C=49^{\circ}39'47.02''$ $b=123.34$ $c=103.068$ $R=67.60763$ $r=32.93385$ $r_1=109.9998$ $r_2=112.73784$ $r_3=80.62681$
18. $\alpha=786.33$ $A=21^{\circ}33'9.62''$ $C=57^{\circ}14'20.48''$	$A=101^{\circ}12'29.90''$ $b=294.48$ $c=674.11$ $R=400.8097$ $r=110.9599$ $r_1=1068.3954$ $r_2=167.0089$ $r_3=478.7944$
19. $\alpha=2358.99$ $A=80^{\circ}10'22.64''$ $B=67^{\circ}33'33.80''$	$C=26^{\circ}10'3.56''$ $b=2155.23$ $c=1040.34$ $R=1182.131$ $r=408.0798$ $r_1=2614.54$ $r_2=1068.843$ $r_3=652.2222$
20. $\alpha=70.48$ $A=28^{\circ}46'16.82''$ $B=67^{\circ}1'50.08''$	$C=84^{\circ}11'53.10''$ $b=134.822$ $c=145.682$ $R=73.21606$ $r=26.9545$ $r_1=45.0119$ $r_2=116.223$ $r_3=158.5638$
21. $\alpha=4322.56$ $A=72^{\circ}19'11.54''$ $C=51^{\circ}21'10.96''$	$B=56^{\circ}19'37.50''$ $b=3775.64$ $c=3543.32$ $R=2268.424$ $r=1094.909$ $r_1=4253.906$ $r_2=3116.297$ $r_3=2798.406$
22. $\alpha=1204.66$ $A=74^{\circ}14'44.72''$ $C=37^{\circ}20'44.36''$	$A=68^{\circ}24'30.92''$ $b=1240.9$ $c=785.92$ $R=647.7839$ $r=281.4538$ $r_1=1100.2722$ $r_2=1225.26$ $r_3=547.0578$
23. $\alpha=187.035$ $A=84^{\circ}46'54.70''$ $B=61^{\circ}58'11.62''$	$C=33^{\circ}14'53.68''$ $b=165.783$ $c=102.972$ $R=93.90667$ $r=37.29846$ $r_1=208.0306$ $r_2=136.8516$ $r_3=68.04294$
24. $\alpha=20077.67$ $A=82^{\circ}31'16.78''$ $C=32^{\circ}1'18.18''$	$B=65^{\circ}27'25.04''$ $b=18420.33$ $c=10737.33$ $R=10124.96$ $r=3982.967$ $r_1=21597.22$ $r_2=15821.55$ $r_3=7064.05$
25. $\alpha=5618.156$ $B=71^{\circ}46'44.56''$ $C=54^{\circ}46'38.52''$	$A=53^{\circ}26'36.92''$ $b=6643.4$ $c=5713.6$ $R=3497.05$ $r=1696.249$ $r_1=4524.564$ $r_2=6503.416$ $r_3=4656.465$
26. $\alpha=152.294$ $A=60^{\circ}58'15.38''$ $B=78^{\circ}51'16.72''$	$C=40^{\circ}10'27.9''$ $b=170.89$ $c=112.3631$ $R=87.08746$ $r=38.54804$ $r_1=128.2040$ $r_2=179.0552$ $r_3=79.63866$
27. $\alpha=7710.144$ $B=51^{\circ}22'32.94''$ $C=63^{\circ}34'12.98''$	$A=65^{\circ}3'14.08''$ $b=6643.4$ $c=7614.7$ $R=4251.738$ $r=2087.92$ $r_1=7004.926$ $r_2=5283.446$ $r_3=6806.502$

EXAMPLES. V.—Continued.

DATA.		ANSWERS.	
28.	$a=854.45$ $b=61^{\circ}45'17.76''$ $c=40^{\circ}45'14.78''$	$A=77^{\circ}29'27.46''$ $B=771.0144$ $C=571.36$ $R=437.6132$	$r=195.7687$ $r_1=881.4266$ $r_2=656.7996$ $r_3=407.9952$
29.	$a=1999.17$ $B=60^{\circ}46'54''$ $C=52^{\circ}6'26.74''$	$A=67^{\circ}3'39.26''$ $b=1894.38$ $c=1714.08$ $R=1085.274$	$r=533.2875$ $r_1=1857.858$ $r_2=1044.306$ $r_3=1372.217$
30.	$a=856.41$ $A=75^{\circ}13'54.42''$ $B=59^{\circ}17'26.38''$	$C=45^{\circ}28'39.2''$ $b=761.47$ $c=631.40$ $R=442.8341$	$r=206.7065$ $r_1=816.0095$ $r_2=640.079$ $r_3=471.3543$

VI.—Triangles in which three sides are given.

First find r the radius of the inscribed circle by the formula $r = \sqrt{\frac{s(s-a)(s-b)(s-c)}{s}}$; then the three half angles from the equations $\tan \frac{1}{2}A = \frac{r}{s-a}$, $\tan \frac{1}{2}B = \frac{r}{s-b}$, $\tan \frac{1}{2}C = \frac{r}{s-c}$, correcting the results so as to make the sum of the three half angles exactly 90° . In general this will not involve alterations beyond $\pm .02''$; should greater changes be required, the work must be suspected. The values of r_1 , r_2 , r_3 may then be obtained from the equations $r_1 = \tan \frac{1}{2}A$, $r_2 = \tan \frac{1}{2}B$, $r_3 = \tan \frac{1}{2}C$, and $R = \frac{1}{r_1}(r_1 + r_2 + r_3 - r)$. Should a further check be desired, $2R$ may be calculated from one of the three $\frac{a}{\sin A}$, $\frac{b}{\sin B}$, $\frac{c}{\sin C}$, taking the angle nearest 45° . An example is worked out below.

DATA.		
$a=6634.39$	$s-a=1426.719$	$\log s-a=3.1543384$
$b=5646.67$	$s-b=2414.439$	$\log s-b=3.3828162$
$c=3841.158$	$s-c=4219.951$	$\log s-c=3.6253075$
$2s=16122.218$	$s=8061.109$	$-\log s=3.9063948$
$s=8061.109$		$2 \log r=6.2560673$
		$\log r=3.1280337$

$L \tan \frac{1}{2}A$	$L \tan \frac{1}{2}B$	$L \tan \frac{1}{2}C$
13.1280337	13.1280337	13.1280337
3.1543384	3.3828162	3.6253075
9.9736953	9.7452175	9.5027262
$43^{\circ}16' - \left. \begin{array}{l} 7073 \\ 2532 \end{array} \right\} \frac{7073}{120}$	$29^{\circ}59' - \left. \begin{array}{l} 403 \\ 2975 \end{array} \right\} \frac{403}{228}$	$17^{\circ}39' + \left. \begin{array}{l} 6721 \\ 4371 \end{array} \right\} \frac{6721}{541}$
7200(2.84)	13680(4.6)	32460(7.43)
2136	1780	1863
110		115

$\frac{1}{2}A=43^{\circ}15'57.16''$	$\log r=3.1280337$	1.9736953	1.7452175	1.5027262
$\frac{1}{2}B=29^{\circ}4'55.40''$	113	3.9063948	3.9063948	3.9063948
$\frac{1}{2}C=17^{\circ}39'7.44''$	224	$\log r_1=3.8800901$	$\log r_2=3.6516123$	$\log r_3=3.4091210$
		873	075	2
$90^{\circ}0'0''$	<u>1342.87</u>	<u>7587.348</u>	<u>4483.449</u>	<u>2565.199</u>
		28	48	-2
$-r=-1342.87$	$\log c=3.5844621$	1784		
$r_1=7587.348$	$-\log \sin C=1.7618208$	15-446	$A=86^{\circ}31'54.32''$	$r_1=7587.348$
$r_2=4483.449$	442	-12=-3	$B=58^{\circ}9'50.8''$	$r_2=4483.449$
$r_3=2565.199$	$\log 2R=3.8225971$	30	$C=35^{\circ}18'14.88''$	$r_3=2565.199$
<u>$4R=13293.126$</u>	<u>6646.562</u>	41	$r=1342.87$	<u>$R=3323.281$</u>

EXAMPLES. VI.

DATA.

ANSWERS.

1. $a=159928.8$ $b=108792.1$ $c=183626.3$	$A=59^{\circ}59'59.8''$ $B=36^{\circ}5'39.6''$ $C=83^{\circ}54'20.6''$ $r=28240.42$	$r_1=130581.3$ $r_2=73694.15$ $r_3=203310.9$ $R=92334.98$	
2. $a=8175.801$ $b=9387.248$ $c=5561.611$	$A=60^{\circ}$ $B=83^{\circ}54'20.5''$ $C=36^{\circ}5'39.5''$ $r=1955.214$	$r_1=6675.515$ $r_2=10393.55$ $r_3=3707.335$ $R=4720.299$	$\Delta_2=3825.637$ $=b-c$
3. $a=9943.485$ $b=8660.254$ $c=5891.161$	$A=83^{\circ}54'20.5''$ $B=60^{\circ}$ $C=36^{\circ}5'39.5''$ $r=2071.068$	$r_1=11000.415$ $r_2=7071.068$ $r_3=3990.584$ $R=5000$	$\Delta_1=2071.067$ $\Delta_2=4052.325$ $\Delta_3=2071.067$
4. $a=28466.08$ $b=24792.47$ $c=16865.15$	$A=83^{\circ}54'20.4''$ $B=60^{\circ}$ $C=36^{\circ}5'39.6''$ $r=5929.031$	$r_1=31517.59$ $r_2=20242.97$ $r_3=11424.21$ $R=14313.93$	$\Delta_1=5929.008$ $\Delta_2=11600.93$ $\Delta_3=5929.019$
5. $a=2356.186$ $b=2365.324$ $c=1383.01$	$A=72^{\circ}36'20.57''$ $B=73^{\circ}19'40.76''$ $C=34^{\circ}3'52.67''$ $r=511.3707$	$r_1=2242.342$ $r_2=2272.171$ $r_3=935.0524$ $R=1234.549$	
6. $a=38714.69$ $b=33539.37$ $c=28471.26$	$A=76^{\circ}46'57.14''$ $B=57^{\circ}29'52.58''$ $C=45^{\circ}43'10.28''$ $r=9229.178$	$r_1=39904.455$ $r_2=27628.73$ $r_3=21232.35$ $R=19884.09$	
7. $a=468.2137$ $b=331.2970$ $c=258.9733$	$A=104^{\circ}18'19.36''$ $B=43^{\circ}17'8.40''$ $C=32^{\circ}24'32.24''$ $r=78.54318$	$r_1=681.13255$ $r_2=209.9995$ $r_3=153.8038$ $R=241.5982$	
8. $a=2846.608$ $b=2479.247$ $c=1687.515$	$A=83^{\circ}53'9.77''$ $B=59^{\circ}59'46.97''$ $C=36^{\circ}7'3.26''$ $r=593.149$	$r_1=3151.134$ $r_2=2024.438$ $r_3=1143.37$ $R=1431.446$	$\Delta_1=592.3827$ $\Delta_2=592.4747$ $\Delta_3=1159.2203$

EXAMPLES. VI.—Continued.

DATA.		ANSWERS.		
9.	$a=47674.27$ $b=46639.05$ $c=33611.96$	$A=70^{\circ}46'47.08''$ $B=67^{\circ}28'53.44''$ $C=41^{\circ}44'19.48''$ $r=11571.21$	$r_1=45438.86$ $r_2=42723.55$ $r_3=24385.79$ $R=25244.25$	$\Delta_1=7284.168$
10.	$a=466.3904$ $b=456.2141$ $c=270.8213$	$A=75^{\circ}0'0''$ $B=70^{\circ}52'57.7''$ $C=34^{\circ}7'2.3''$ $r=100$	$r_1=457.8737$ $r_2=424.7102$ $r_3=183.1016$ $R=241.4214$	$\Delta_1=100$ $\Delta_2=159.8853$ $\Delta_3=60.32828$
11.	$a=19674.04$ $b=15361.02$ $c=9837.02$	$A=100^{\circ}21'30.58''$ $B=50^{\circ}10'45.12''$ $C=29^{\circ}27'44.3''$ $r=3312.615$	$r_1=26908.75$ $r_2=10504.845$ $r_3=5899.026$ $R=10000$	
12.	$a=9682.458$ $b=9990.51$ $c=2919.436$	$A=75^{\circ}31'20.68''$ $B=87^{\circ}30'11.67''$ $C=16^{\circ}58'27.65''$ $r=1250$	$r_1=8749.988$ $r_2=10814.376$ $r_3=1685.044$ $R=5000$	
13.	$a=47674.27$ $b=46639.05$ $c=33611.96$	$A=70^{\circ}46'47.08''$ $B=67^{\circ}28'53.44''$ $C=41^{\circ}44'19.48''$ $r=11571.21$	$r_1=45438.76$ $r_2=42723.55$ $r_3=24385.78$ $R=25244.25$	
14.	$a=375.3259$ $b=312.7715$ $c=271.0686$	$A=79^{\circ}39'36.48''$ $B=55^{\circ}3'53.24''$ $C=45^{\circ}16'30.28''$ $r=86.95647$	$r_1=400$ $r_2=250$ $r_3=200$ $R=190.7608$	
15.	$a=14538.87$ $b=34045.37$ $c=30139.84$	$A=24^{\circ}42'1.5''$ $B=95^{\circ}16'15.9''$ $C=60^{\circ}1'42.6''$ $r=5500.802$	$r_1=8684.138$ $r_2=43489.62$ $r_3=22312.04$ $R=17396.25$	
16.	$a=7406.25$ $b=31560.45$ $c=30137.34$	$A=13^{\circ}32'3.74''$ $B=94^{\circ}13'58.62''$ $C=72^{\circ}13'57.64''$ $r=3221.169$	$r_1=4100.008$ $r_2=37203.835$ $r_3=25210.84$ $R=15823.38$	
17.	$a=10927.86$ $b=8517.08$ $c=3642.62$	$A=122^{\circ}47'45.58''$ $B=40^{\circ}55'54.4''$ $C=16^{\circ}16'20.02''$ $r=1129.583$	$r_1=21171.03$ $r_2=4308.21$ $r_3=1650.345$ $R=6500$	
18.	$a=5761.367$ $b=5017.848$ $c=3413.403$	$A=83^{\circ}54'20.57''$ $B=60^{\circ}0'0''$ $C=36^{\circ}5'39.43''$ $r=1200$	$r_1=6378.981$ $r_2=4097.055$ $r_3=2312.188$ $R=2897.056$	$\Delta_1=\Delta_2=r$ $\Delta_3=2347.964$ $=a-c$
19.	$a=94371.5$ $b=124351.6$ $c=106381.7$	$A=47^{\circ}30'$ $B=76^{\circ}17'14.52''$ $C=56^{\circ}12'45.48''$ $r=30000.31$	$r_1=71524.76$ $r_2=127657.63$ $r_3=86817.93$ $R=64000$	$\Delta_1=15998.76$
20.	$a=4157.855$ $b=4812.793$ $c=3608.528$	$A=57^{\circ}2'28''$ $B=76^{\circ}13'21.6''$ $C=46^{\circ}44'10.4''$ $r=1158.427$	$r_1=3417.89$ $r_2=4933.676$ $r_3=2717.593$ $R=2477.683$	

EXAMPLES. VI.--Continued.

DATA.		ANSWERS.	
21.	$a=9401.45$ $b=7360.12$ $c=5318.79$	The sides in $A, P.$, and the greatest angle exceeds the least by 60° . $8r=3A$.	$A=94^\circ 20' 27.92''$ $B=51^\circ 19' 4.16''$ $C=34^\circ 20' 27.92''$ $r=1767.844$ $r_1=11910.02$ $r_2=5303.53$ $r_3=3411.289$ $A=4714.249$ $\Delta_1=2357.125$
22.	$a=2692538$ $b=1860498$ $c=1028458$		$A=135^\circ 31' 20.04''$ $B=28^\circ 57' 18.08''$ $C=15^\circ 31' 20.98''$ $r=240189.2$ $r_1=6125317$ $r_2=720507.9$ $r_3=380361.9$ $A=1921514$ $\Delta_1=1664081$
23.	$a=964.701$ $b=955.078$ $c=945.455$	The sides in $A, P.$, $A-C=2^\circ$.	$A=61^\circ 0' 18.14''$ $B=59^\circ 59' 23.70''$ $C=59^\circ 0' 18.10''$ $r=275.6513$ $r_1=843.9606$ $r_2=820.9536$ $r_3=810.6189$ $A=551.4705$ $\Delta_1=9.622469$
24.	$a=2089.88$ $b=2413.68$ $c=1206.96$		$A=59^\circ 58' 46.83''$ $B=90^\circ 1' 1.35''$ $C=30^\circ 0' 11.82''$ $r=441.7114$ $r_1=1647.81$ $r_2=2856.109$ $r_3=765.1524$ $A=1206.84$ $\Delta_1=624.7486$ $\Delta_2=1208.062$ $\Delta_3=625.855$
25.	$a=4794.217$ $b=4329.393$ $c=3655.232$		$A=73^\circ 15' 10.41''$ $B=59^\circ 51' 12.13''$ $C=46^\circ 53' 37.40''$ $r=1185.846$ $r_1=4749.779$ $r_2=3078.041$ $r_3=2771.157$ $A=2503.283$
26.	$a=544.36$ $b=426.39$ $c=425.09$		$A=79^\circ 28' 50.42''$ $B=50^\circ 21' 53.56''$ $C=50^\circ 9' 10.02''$ $r=127.6707$ $r_1=580.2548$ $r_2=328.155$ $r_3=326.5914$ $A=276.8326$
27.	$a=51509.95$ $b=72027.91$ $c=63926$		$A=44^\circ 1' 38.62''$ $B=76^\circ 22' 16.08''$ $C=59^\circ 36' 5.3''$ $r=17070.53$ $r_1=37896.22$ $r_2=37721.52$ $r_3=53682.39$ $A=37057.2$
28.	$a=10301.99$ $b=10825.782$ $c=9205.4$		$A=61^\circ 16' 11.50''$ $B=67^\circ 8' 35.18''$ $C=51^\circ 35' 13.32''$ $r=2880.918$ $r_1=8981.956$ $r_2=10065.804$ $r_3=7329.696$ $A=5874.135$
29.	$a=5844.305$ $b=7202.791$ $c=4602.7$		$A=54^\circ 4' 4.80''$ $B=86^\circ 18' 43.14''$ $C=39^\circ 37' 12''$ $r=1520.916$ $r_1=4503.11$ $r_2=8274.385$ $r_3=3178.898$ $A=3608.869$
30.	$a=1168.861$ $b=1082.5782$ $c=1278.52$		$A=58^\circ 38' 46.14''$ $B=52^\circ 16' 21.68''$ $C=60^\circ 4' 52.18''$ $r=334.8419$ $r_1=991.3952$ $r_2=806.0432$ $r_3=1214.8782$ $A=684.3687$
31.	$a=1262.92$ $b=1332.60$ $c=1142.72$		$A=60^\circ 46' 54.01''$ $B=67^\circ 3' 39.26''$ $C=52^\circ 9' 26.73''$ $r=355.525$ $r_1=1096.2042$ $r_2=1238.5722$ $r_3=914.8112$ $A=723.5157$
32.	$a=18943.8$ $b=25692.3$ $c=22844.1$		$A=45^\circ 28' 39.20''$ $B=75^\circ 13' 54.42''$ $C=59^\circ 17' 26.38''$ $r=6201.195$ $r_1=14140.63$ $r_2=25998.28$ $r_3=19202.37$ $A=13285.02$

EXAMPLES. VI.--Continued.

DATA.		ANSWERS.		
33.	$a=70137$ $b=66630$ $c=76147$	$A=58^{\circ}23'20.49''$ $B=54^{\circ}09.25'$ $C=71^{\circ}36'30.26''$ $r=20293.99$	$r_1=59483.42$ $r_2=54245.50$ $r_3=71278.06$ $K=41178.20$	
34.	$a=3506.8$ $b=4282.05$ $c=2856.8$	$A=54^{\circ}31'24.61''$ $B=83^{\circ}55'0.40''$ $C=41^{\circ}33'34.99''$ $r=935.773$	$r_1=2742.83$ $r_2=4785.72$ $r_3=2019.82$ $K=2153.15$	
35.	$a=786.934$ $b=604.81$ $c=431.5957$	$A=97^{\circ}23'42.2''$ $B=49^{\circ}39'21.88''$ $C=32^{\circ}56'55.92''$ $r=141.9714$	$r_1=1037.641$ $r_2=421.7921$ $r_3=269.6064$ $K=396.767$	
36.	$a=786.934$ $b=748.887$ $c=431.5957$	$A=78^{\circ}35'37.44''$ $B=68^{\circ}53'4.20''$ $C=32^{\circ}31'18.36''$ $r=161.04$	$r_1=805.0664$ $r_2=674.625$ $r_3=286.9277$ $K=401.3948$	
37.	$a=43783$ $b=43397$ $c=43012$	$A=60^{\circ}53'9.3''$ $B=59^{\circ}59'29.07''$ $C=50^{\circ}7'21.63''$ $r=12525.76$	$r_1=38257.23$ $r_2=37576.69$ $r_3=36921.59$ $K=25057.44$	$\Delta_1=385.1494$ $\Delta_2=771.168$ $\Delta_3=399.6372$
38.	$a=96.264$ $b=114.735$ $c=106.848$	$A=51^{\circ}20'53.50''$ $B=68^{\circ}33'38.10''$ $C=60^{\circ}5'28.34''$ $r=30.12111$	$r_1=76.39623$ $r_2=108.3302$ $r_3=91.92327$ $K=61.63215$	$\Delta_1=9.255574$
39.	$a=224.626$ $b=218.323$ $c=199.92$	$A=64^{\circ}47'27.34''$ $B=61^{\circ}34'17.14''$ $C=53^{\circ}38'15.52''$ $r=61.428976$	$r_1=203.9497$ $r_2=191.5018$ $r_3=162.4983$ $K=124.1302$	
40.	$a=14724$ $b=27601$ $c=17224$	$A=50^{\circ}0'30.76''$ $B=66^{\circ}10'31.38''$ $C=63^{\circ}39'57.86''$ $r=4687.54$	$r_1=11554.79$ $r_2=16188.95$ $r_3=15380.63$ $K=9609.21$	
41.	$a=63341$ $b=52587$ $c=29448$	$A=97^{\circ}2'17.00''$ $B=55^{\circ}29'1.08''$ $C=27^{\circ}28'41.02''$ $r=10571.92$	$r_1=82214.21$ $r_2=38229.75$ $r_3=17771.81$ $K=31910.96$	
42.	$a=1266.82$ $b=704.04$ $c=1166.82$	$A=81^{\circ}9'58.50''$ $B=33^{\circ}18'34.24''$ $C=65^{\circ}31'27.26''$ $r=258.7082$	$r_1=1343.8572$ $r_2=469.3244$ $r_3=1009.5812$ $K=641.0137$	
43.	$a=1750.23$ $b=1577.61$ $c=1033.44$	$A=81^{\circ}17'39.16''$ $B=62^{\circ}59'51.52''$ $C=35^{\circ}42'29.32''$ $r=399.5202$	$r_1=1872.1467$ $r_2=336.2372$ $r_3=702.3975$ $K=885.3153$	
44.	$a=13467.87$ $b=8287.92$ $c=7251.93$	$A=120^{\circ}$ $B=32^{\circ}12'15.22''$ $C=27^{\circ}47'44.78''$ $r=1794.387$	$r_1=25121.42$ $r_2=4186.903$ $r_3=3588.774$ $K=7775.68$	$\Delta_2=15539.85$ $=b+c$

VII.—Triangles in which two sides and the included angle are given.

Let b, c be the two given lengths ($b > c$), A the included angle, then we find $\frac{1}{2}(B - C)$ by the equation $\tan \frac{1}{2}(B - C) = \frac{b - c}{b + c} \cot \frac{1}{2}A$; and $\frac{1}{2}(B + C) = 90^\circ - \frac{A}{2}$; $B = \frac{1}{2}(B + C) + \frac{1}{2}(B - C)$, $C = \frac{1}{2}(B + C) - \frac{1}{2}(B - C)$. The value of $2R$ is then calculated from each of the expressions $b/\sin B, c/\sin C$, which should agree ($p.p.$). The formulae $a = 2R \sin A$, $r = (s - a) \tan \frac{A}{2}$, $r_1 = s \tan \frac{A}{2}$, $r_2 = (s - c) \cot \frac{1}{2}A$, $r_3 = (s - b) \cot \frac{1}{2}A$ enable us to find a, r, r_1, r_2, r_3 ; and the equation $4R = r_1 + r_2 + r_3 - r$ supplies another check.

DATA.		EXAMPLE WORKED OUT.	
$b=33417$	$b-c=9990$	$I \cot \frac{1}{2}A=10 \ 5228579$	4592
$c=23427$	$b+c=56844$	$\log(b-c)=3 \ 9995655$	$10''=765 \ 3$
$A=33^\circ 23' 37''$	$A=16^\circ 41' 48.5''$	$-\log(b+c)=-4 \ 7546816$	$1.5=114 \ 8$
		$I \tan \frac{1}{2}(B - C)=9 \ 678268$	$4560(1.58$
		$30''22'-$	344
		$2896 \ 7$	1664
			216
$\frac{1}{2}(B - C)=30^\circ 21' 58.42''$		$\log b=4 \ 5239675$	$308=5+10+10$
$\frac{1}{2}(B + C)=73^\circ 18' 11.5''$		$-\log \sin B=-1 \ 9874955$	$250 \ 4$
$B=103^\circ 40' 9.92''$	$180^\circ - B=76^\circ 19' 50.08''$	$\log 2R=4 \ 5364463$	$5 \ 7$
$C=42^\circ 56' 13.08''$		48	
		34391.12	15
$\log c=4.3697167$	1358	$\log 2R=4.5364463$	1916
$-\log \sin C=1.8332408$	$12-271 \ 6$	$\log \sin A=1.7405505$	$36=1149 \ 6$
$296 \ 7$	$1.2=27 \ 2$	$1181 \ 1$	$1=31 \ 9$
$\log 2R=4.5364463$	$1.2=-2 \ 7$	$\log a=4.2771149$	
		047	
		18928.44	102
$\log \cot \frac{1}{2}A=0.5229459$	$a=18928.44$	$s-a=18957.78$	$\log(s-a)=4.2777877$
$\log \tan \frac{1}{2}A=1.4770541$	$b=33417$	$s-b=4409.22$	$\log(s-b)=3.6502317$
	$c=23427$	$s-c=14459.22$	$\log(s-c)=4.1601449$
	75772.44	$s=37886.22$	$\log s=4.5784813$
	$s=37886.22$		
$\log r$	$\log r_1$	$\log r_2$	$\log r_3$
1.4770541	1.4770541	$.5229459$	$.5229459$
4.2777877	4.5784813	4.1601449	3.6502317
3.7548418	4.0555354	4.6830908	4.1731776
374	12	831	571
5686.457	11364.11	48204.86	11899.70
44	42	77	250

ANSWERS.

$B=103^\circ 40' 9.92''$	$a=18928.44$	$r=5686.457$	$r_2=48204.86$
$C=42^\circ 56' 13.08''$	$R=17195.56$	$r_1=11364.11$	$r_3=14899.70$

EXAMPLES. VII.

DATA.

ANSWERS.

1. $b=23571.26$ $c=20753.82$ $A=67^{\circ}31'55''$		$B=61^{\circ}39'53.64''$ $C=50^{\circ}48'11.36''$ $a=24747.05$ $R=13389.91$	$r=6544.85$ $r_1=23090.21$ $r_2=20614.16$ $r_3=16400.11$	
2. $b=8438.715$ $c=4219.357$ $A=59^{\circ}37'56.4''$		$B=90^{\circ}22'6.07''$ $C=29^{\circ}59'57.53''$ $a=7281.067$ $R=4219.444$	$r=1540.727$ $r_1=5713.363$ $r_2=10033.872$ $r_3=2671.274$	
3. $b=378.9217$ $c=269.1351$ $A=88^{\circ}59'2.4''$		$B=55^{\circ}17'30.72''$ $C=35^{\circ}43'26.88''$ $a=460.8674$ $R=230.4699$	$r=91.94954$ $r_1=544.71625$ $r_2=200.4319$ $r_3=178.6812$	
4. $b=13347.61$ $c=11258.73$ $A=47^{\circ}31'57''$		$B=77^{\circ}8'44''$ $C=55^{\circ}19'19''$ $a=10099.1$ $R=6845.362$	$r=3194.125$ $r_1=7641.255$ $r_2=13839$ $r_3=9095.308$	
5. $b=99144.49$ $c=60876.14$ $A=45^{\circ}$		$B=97^{\circ}30'$ $C=37^{\circ}30'$ $a=70710.68$ $R=50000$	$r=18496.69$ $r_1=47786.01$ $r_2=131549.34$ $r_3=39161.36$	
6. $b=5.6794$ $c=5.6318$ $A=44^{\circ}59'6.26''$		$B=68^{\circ}5'23.27''$ $C=66^{\circ}55'30.47''$ $a=4.327479$ $R=3.060782$	$r=1.445844$ $r_1=3.237680$ $r_2=5.283127$ $r_3=5.168167$	
7. $b=87.92154$ $c=29.30718$ $A=70^{\circ}31'43.62''$		$B=90^{\circ}$ $C=19^{\circ}28'16.38''$ $a=82.89322$ $R=43.96077$	$r=12.13143$ $r_1=70.75379$ $r_2=100.06097$ $r_3=17.16775$	
8. $b=99905.1$ $c=29194.36$ $A=75^{\circ}31'20.7''$		$B=87^{\circ}30'11.65''$ $C=16^{\circ}58'27.65''$ $a=96824.54$ $R=50000$	$r=12500$ $r_1=87499.9$ $r_2=108143.67$ $r_3=16856.4$	
9. $b=1243.516$ $c=1063.817$ $A=47^{\circ}30'$		$B=76^{\circ}17'14.56''$ $C=56^{\circ}12'45.44''$ $a=943.715$ $R=640$	$r=300.0031$ $r_1=715.2478$ $r_2=1276.5762$ $r_3=868.1792$	$\Delta_1=159.9976$ $\Delta_2=343.1666$ $\Delta_3=185.617$
10. $b=57777.78$ $c=47222.22$ $A=129^{\circ}18'27.52''$	$180r=9r_1=76r_2=95r_3$	$B=28^{\circ}4'21.01''$ $C=22^{\circ}37'11.47''$ $a=95000$ $R=61388.89$	$r=10555.56$ $r_1=211111.1$ $r_2=25000$ $r_3=20000$	
11. $b=5237.828$ $c=6673.472$ $A=87^{\circ}50'57.8''$		$B=38^{\circ}56'32.8''$ $C=53^{\circ}12'29.4''$ $a=8327.466$ $R=4166.667$	$r=1725.888$ $r_1=9746.503$ $r_2=3577.745$ $r_3=5068.313$	$\Delta_1=1725.894$
12. $b=14045.86$ $c=8109.38$ $A=30^{\circ}$		$B=120^{\circ}$ $C=30^{\circ}$ $a=8109.38$ $R=8109.38$	$r=1881.79$ $r_1=4054.69$ $r_2=26209.93$ $r_3=4054.69$	

EXAMPLES. VII.—Continued.

DATA.	ANSWERS.
13. $b=1404.59$ $c=810.94$ $A=30^{\circ}0'0.07''$	$B=119^{\circ}59'59.965''$ $C=29^{\circ}59'59.965''$ $a=810.941$ $R=810.9405$ $r=188.1794$ $r_1=405.4705$ $r_2=2621.0007$ $r_3=405.4702$
14. $b=4568.39$ $c=864.29$ $A=110^{\circ}34'11.96''$	$B=60^{\circ}$ $C=9^{\circ}25'48.04''$ $a=4938.8$ $R=2637.561$ $r=356.427$ $r_1=7484.971$ $r_2=2993.989$ $r_3=427.712$ $\Delta_1=2253.115$ $\Delta_2=4074.51$ $\Delta_3=2253.115$
15. $b=9237.97$ $c=6598.55$ $A=81^{\circ}47'12.46''$	$B=60^{\circ}$ $C=38^{\circ}12'47.54''$ $a=10557.08$ $R=5333.545$ $r=2285.805$ $r_1=11429.025$ $r_2=7014.35$ $r_3=4571.61$ $\Delta_1=2015.891$ $\Delta_2=3959.131$ $\Delta_3=2015.891$
16. $b=47898.32$ $c=39817.65$ $A=65^{\circ}26'23.6''$	$B=65^{\circ}26'23.6''$ $C=49^{\circ}7'12.8''$ $a=47898.32$ $R=26331.48$ $r=12791.04$ $r_1=43564.76$ $r_2=43564.76$ $r_3=30987.44$
17. $b=48127.93$ $c=36085.28$ $A=57^{\circ}2'28''$	$B=76^{\circ}13'21.62''$ $C=46^{\circ}44'10.38''$ $a=41578.55$ $R=24776.83$ $r=11584.27$ $r_1=34178.9$ $r_2=49336.76$ $r_3=27175.93$
18. $b=4157.855$ $c=3608.528$ $A=76^{\circ}13'21.6''$	$B=57^{\circ}2'28''$ $C=46^{\circ}44'10.4''$ $a=4812.793$ $R=2477.683$ $r=1158.427$ $r_1=4933.676$ $r_2=3417.89$ $r_3=2717.593$
19. $b=23571.26$ $c=20753.52$ $A=67^{\circ}31'55''$	$B=61^{\circ}39'55.86''$ $C=50^{\circ}48'9.14''$ $a=24746.85$ $R=13389.84$ $r=6544.721$ $r_1=23090.03$ $r_2=20614.22$ $r_3=16399.73$
20. $b=30901.7$ $c=58778.53$ $A=18^{\circ}$	$B=18^{\circ}$ $C=144^{\circ}$ $a=30901.7$ $R=50000$ $r=4654.803$ $r_1=9549.152$ $r_2=9549.152$ $r_3=185556.5$
21. $b=12369.12$ $c=4123.04$ $A=40^{\circ}55'55''$	The triangle in which the greatest side is 3 times the least, and the greatest angle 3 times the mean. <i>q.p.</i> $B=122^{\circ}47'44.84''$ $C=16^{\circ}16'20.16''$ $a=9640.399$ $R=7357.262$ $r=1278.566$ $r_1=4876.439$ $r_2=23963.16$ $r_3=1868.021$ $\Delta_1=5942.711$ $\Delta_2=14974.37$ $\Delta_3=9397.907$
22. $b=13571$ $c=8965$ $A=79^{\circ}55'58.5''$	$B=63^{\circ}44'18.57''$ $C=36^{\circ}19'42.93''$ $a=14899.98$ $R=7566.486$ $r=3199.883$ $r_1=15687.6$ $r_2=11636.99$ $r_3=6141.241$ $\Delta_1=2971.191$ $\Delta_2=6021.614$ $\Delta_3=3159.603$
23. $b=18791.69$ $c=11275.05$ $A=68^{\circ}35'57.6''$	$B=75^{\circ}49'39.51''$ $C=35^{\circ}34'22.89''$ $a=18045.27$ $R=9690.8$ $r=4100.197$ $r_1=16409.691$ $r_2=18736.413$ $r_3=7717.292$ $\Delta_1=3800.416$
24. $b=133.7127$ $c=206.3273$ $A=58^{\circ}21'37.4''$	$B=39^{\circ}53'33.16''$ $C=81^{\circ}44'49.44''$ $a=177.4979$ $R=104.2432$ $r=45.384$ $r_1=144.5037$ $r_2=93.9095$ $r_3=223.9436$

EXAMPLES. VII.—Continued.

DATA.		ANSWERS.	
25.	$b=8233.59$ $c=6618.853$ $A=83^{\circ}8'1.8''$	$B=55^{\circ}25'20.65''$ $C=41^{\circ}26'37.55''$ $a=9928.291$ $R=5000.006$	$r=2183.392$ $r_1=10987.9$ $r_2=6508.179$ $r_3=4687.339$
26.	$b=23870.56$ $c=10775.56$ $A=41^{\circ}24'34.6''$	$b+c=2a, B-C=90^{\circ}$ $R=4r, r_1=3r$ $B=114^{\circ}17'42.7''$ $C=24^{\circ}17'42.7''$ $a=17323.06$ $R=13095$	$r=3273.749$ $r_1=9821.247$ $r_2=40239.337$ $r_3=5593.201$
27.	$b=5361.796$ $c=5957.551$ $A=23^{\circ}33'23.3''$	$B=64^{\circ}3'19.87''$ $C=92^{\circ}23'10.83''$ $a=2383.022$ $R=2981.364$	$r=931.6764$ $r_1=1428.571$ $r_2=4285.715$ $r_3=7142.852$
28.	$b=138.74$ $c=116.82$ $A=34^{\circ}33'37''$	$B=88^{\circ}8'4.07''$ $C=57^{\circ}18'18.93''$ $a=78.7452$ $R=69.4068$	$r=27.50223$ $r_1=51.99869$ $r_2=161.7969$ $r_3=91.33386$
29.	$b=1687.48$ $c=2531.22$ $A=104^{\circ}28'39.04''$	$B=28^{\circ}57'18.08''$ $C=46^{\circ}34'2.88''$ $a=3374.96$ $R=1742.822$	$r=544.6318$ $r_1=4901.688$ $r_2=980.3372$ $r_3=1633.896$
30.	$b=31428.54$ $c=35714.25$ $A=69^{\circ}47'25.64''$	$74r=20r_1=30r_2=24r_3$ $B=49^{\circ}52'33.45''$ $C=60^{\circ}20'0.91''$ $a=38571.41$ $R=20550.91$	$r=9964.067$ $r_1=36867.08$ $r_2=24578.06$ $r_3=30722.57$
31.	$b=273.8163$ $c=262.1835$ $A=8^{\circ}13'57.6''$	$B=102^{\circ}39'55.14''$ $C=69^{\circ}6'7.26''$ $a=40.18648$ $R=140.323$	$r=17.8412$ $r_1=20.73332$ $r_2=360.02$ $r_3=198.3801$
32.	$b=24.1368$ $c=12.0696$ $A=59^{\circ}58'48''$	$B=90^{\circ}1'0.17''$ $C=30^{\circ}0'11.83''$ $a=20.89888$ $R=12.068405$	$r=4.417119$ $r_1=16.47823$ $r_2=28.56097$ $r_3=7.651542$
33.	$b=856.41$ $c=761.47$ $A=45^{\circ}28'39.2''$	$B=75^{\circ}13'54.53''$ $C=59^{\circ}17'26.27''$ $a=631.4601$ $R=442.834$	$r=206.7065$ $r_1=471.3542$ $r_2=866.6096$ $r_3=640.079$
34.	$b=786.934$ $c=748.887$ $A=32^{\circ}31'18.36''$	$B=78^{\circ}35'37.44''$ $C=68^{\circ}53'4.20''$ $a=431.5957$ $R=401.3948$	$r=161.04$ $r_1=286.9277$ $r_2=805.0664$ $r_3=674.6250$
35.	$b=3734.67$ $c=3593.19$ $A=52^{\circ}31'49.2''$	$B=65^{\circ}58'31.37''$ $C=61^{\circ}29'39.43''$ $a=3245.25$ $R=2044.444$	$r=1007.332$ $r_1=2608.78$ $r_2=3431.515$ $r_3=3144.813$
36.	$b=487.139$ $c=479.363$ $A=48^{\circ}39'39.6''$	$B=66^{\circ}41'20''$ $C=64^{\circ}36'0.4''$ $a=398.2614$ $R=265.2195$	$r=128.4674$ $r_1=308.5446$ $r_2=448.9992$ $r_3=431.8017$

EXAMPLES. VII.—Continued.

DATA.		ANSWERS.		
37.	$b=433.97$ $c=430.12$ $A=60^{\circ}53'9.3''$	$B=59^{\circ}59'29.06''$ $C=59^{\circ}7'21.64''$ $a=437.83$ $R=250.5744$	$r=125.2576$ $r_1=382.5723$ $r_2=375.7669$ $r_3=309.2159$	$\Delta_1=3.851494$ $\Delta_2=3.996372$ $\Delta_3=7.71168$
38.	$b=1057.81$ $c=1220.55$ $A=27^{\circ}47'44.76''$	$B=60^{\circ}0'0''$ $C=92^{\circ}12'15.24''$ $a=569.59$ $R=610.7271$	$r=211.4054$ $r_1=352.3423$ $r_2=822.1324$ $r_3=1479.839$	$\Delta_1=338.7705$ $\Delta_2=650.9600$ $=c-a$ $\Delta_3=338.7705$

VIII.—Triangles in which two sides and the angle opposite the greater are given.

Let a, b be the two given lengths ($a > b$), and A the given angle. First calculate R by the equation $2R = a/\sin A$; B must be an acute angle and is found from $\sin B = b/2R$; C from $A + B + C = 180^{\circ}$; c from $c = 2R \sin C$. Calculate r, r_1, r_2, r_3 exactly as in V.

EXAMPLE WORKED OUT.

DATA.					
$a=4812.793$	$\log a=3.6823972$	310	$\log b=3.5573301$		
$b=3608.528$	$-\log \sin A=1.9873103$	18 93	$-\log 2R=3.6950757$		
$A=76^{\circ}13'21.6''$	102	3.6=18 6	$L \sin B=9.8622544$	12360(10.4"	
	$\log 2R=3.6950757$	101 6	$46^{\circ}44' + \}$	338	480
	00		1188	206	
	<u>4955.366</u>	57			
$A=76^{\circ}13'21.6''$	$\log 2R=3.6950757$	819	$A=38^{\circ}6'40.8''$		
$B=46^{\circ}44'10.4''$	$\log \sin C=1.9237554$	24=327 6	$\log \tan \frac{A}{2}=1.8943715$	2602	
$C=57^{\circ}2'28.0''$	382	4- 54 6	1769	36=1561 2	
	$\log c=3.6188693$		$=1.8945484$	4.8=208 2	
	36		$\log \cot \frac{A}{2}=0.1054516$		
	<u>4157.855</u>	57			
$a=4812.793$	$s-a=1476.795$	$\log -3.1693202$			
$b=3608.528$	$s-b=2681.06$	$\log -3.4283065$			
$c=4157.855$	$s-c=2131.733$	$\log -3.3287329$			
$2s=12579.176$	$s=6289.588$	$\log =3.7986222$			
$s=6289.588$					
$\log r$	$\log r_1$	$\log r_2$	$\log r_3$		
3.1693202	3.7986222	3.3287329	3.4283065	$-r=-1158.427$	
1.8945484	1.8945484	.1054516	.1054516	$r_1=4933.676$	
3.0638686	3.6931706	3.4341845	3.5337581	$r_2=2717.593$	
585	639	696	466	$r_3=3417.89$	
1158.427	4933.676	2717.593	3417.89	$4R=9910.732$	
101	67	149	115		

ANSWERS.

$R=2477.683$	$C=57^{\circ}2'28''$	$r=1158.427$	$r_2=2717.593$
$B=46^{\circ}44'10.4''$	$c=4157.855$	$r_1=4933.676$	$r_3=3417.89$

EXAMPLES VIII.

DATA.	ANSWERS.
1. $a=266.0007$ $b=138.7214$ $A=62^{\circ}27'27.4''$	$R=150$ $B=27^{\circ}32'32.6''$ $C=90^{\circ}$ $c=300$ $r=52.36105$ $r_1=213.63965$ $r_2=86.36035$ $r_3=352.36105$ $\Delta_1=82.41167$
2. $a=2620.87$ $b=1513.16$ $A=120^{\circ}$	$R=1513.16$ $B=30^{\circ}$ $C=30^{\circ}$ $c=1513.16$ $r=351.13$ $r_1=4890.611$ $r_2=756.5801$ $r_3=756.5801$ $\Delta_1=1107.84$
3. $a=82893.16$ $b=29307.18$ $A=70^{\circ}31'43.55''$	$R=43960.735$ $B=19^{\circ}28'16.45''$ $C=90^{\circ}$ $c=87921.47$ $r=12139.435$ $r_1=70753.725$ $r_2=17167.745$ $r_3=100060.905$ $\Delta_1=29414.78$
4. $a=2731.843$ $b=1600.277$ $A=72^{\circ}58'7.5''$	$R=14285.715$ $B=34^{\circ}3'45''$ $C=72^{\circ}58'7.5''$ $c=2731.843$ $r=591.7336$ $r_1=2612.039$ $r_2=1081.942$ $r_3=2612.039$ $\Delta_1=591.7373$
5. $a=27442.1$ $b=11366.89$ $A=67^{\circ}30'$	$R=14851.56$ $B=22^{\circ}30'$ $C=90^{\circ}$ $c=29703.12$ $r=4552.935$ $r_1=22889.165$ $r_2=6813.955$ $r_3=34250.055$ $\Delta_1=11629.12$
6. $a=968.2454$ $b=291.9436$ $A=75^{\circ}31'20.68''$	$R=500$ $B=16^{\circ}58'27.68''$ $C=87^{\circ}30'11.64''$ $c=999.05114$ $r=125.0001$ $r_1=874.9987$ $r_2=168.564$ $r_3=1081.4372$ $\Delta_1=353.5534$
7. $a=24150.16$ $b=22202.21$ $A=57^{\circ}41'57.5''$	$R=14285.72$ $B=50^{\circ}50'37.48''$ $C=71^{\circ}18'25.01''$ $c=27064.26$ $r=6918.089$ $r_1=20222.015$ $r_2=17506.506$ $r_3=26332.425$ $\Delta_1=2534.167$
8. $a=8571.427$ $b=7142.856$ $A=73^{\circ}44'23.3''$	$R=1464.286$ $B=53^{\circ}7'18.35''$ $C=53^{\circ}7'48.35''$ $c=7142.856$ $r=2412.857$ $r_1=8571.427$ $r_2=5714.284$ $r_3=5714.284$ $\Delta_1=892.8586$
9. $a=673.1282$ $b=336.5642$ $A=92^{\circ}27'53.4''$	$R=336.8757$ $B=29^{\circ}53'36.39''$ $C=57^{\circ}38'30.21''$ $c=569.13$ $r=121.3962$ $r_1=824.1232$ $r_2=211.003$ $r_3=433.7731$
10. $a=47124.88$ $b=35623.91$ $A=65^{\circ}31'48.6''$	$R=25867.18$ $B=43^{\circ}31'7.76''$ $C=70^{\circ}51'3.64''$ $c=48871.84$ $r=12048.9$ $r_1=42436.38$ $r_2=26268.18$ $r_3=46813.06$
11. $a=9401.45$ $b=7360.12$ $A=94^{\circ}20'27.9''$	$R=4714.249$ $B=51^{\circ}19'4.12''$ $C=34^{\circ}20'27.98''$ $c=5318.792$ $r=1767.845$ $r_1=11910.022$ $r_2=5303.53$ $r_3=3411.291$ $\Delta_1=2357.121$
12. $a=7791.368$ $b=5387.123$ $A=69^{\circ}31'48.12''$	$R=4158.254$ $B=40^{\circ}12'23.22''$ $C=70^{\circ}15'48.66''$ $c=7827.959$ $r=1882.335$ $r_1=7290.425$ $r_2=3854.223$ $r_3=7370.705$

EXAMPLES VIII.—Continued.

DATA.		ANSWERS.	
13.	$a=45217.25$ $b=22608.62$ $A=45^{\circ}40'24.6''$	$R=31604.12$ $B=19^{\circ}40'52.88''$ $C=114^{\circ}38'42.52''$ $c=57450.47$	$r=8087.68$ $r_1=25676.23$ $r_2=11174.34$ $r_3=97653.6$
14.	$a=967.8191$ $b=559.4229$ $A=75^{\circ}25'30''$	$R=500$ $B=34^{\circ}0'58.6''$ $C=70^{\circ}33'31.4''$ $c=942.983$	$r=206.6802$ $r_1=955.0335$ $r_2=377.801$ $r_3=873.846$
15.	$a=8045.638$ $b=4257.129$ $A=82^{\circ}37'42.62''$	$R=4056.343$ $B=31^{\circ}39'5.01''$ $C=65^{\circ}43'12.37''$ $c=7395.099$	$r=1585.028$ $r_1=8656.84$ $r_2=2791.74$ $r_3=6361.823$
16.	$a=2428.571$ $b=2142.857$ $A=58^{\circ}59'50.2''$	$R=1416.667$ $B=50^{\circ}42'25.9''$ $C=70^{\circ}17'43.9''$ $c=2667.425$	$r=670.2642$ $r_1=2097.227$ $r_2=1691.417$ $r_3=2548.286$
17.	$a=576.1367$ $b=501.7848$ $A=83^{\circ}54'20.57''$	$R=289.7056$ $B=60^{\circ}0'0''$ $C=36^{\circ}5'39.43''$ $c=341.3403$	$r=120$ $r_1=637.898$ $r_2=409.7056$ $r_3=231.2187$
18.	$a=241.368$ $b=120.696$ $A=90^{\circ}1'1.35''$	$R=120.684$ $B=30^{\circ}0'11.88''$ $C=59^{\circ}58'46.77''$ $c=208.988$	$r=44.17114$ $r_1=285.6109$ $r_2=76.51524$ $r_3=164.781$
19.	$a=4562.141$ $b=2708.213$ $A=70^{\circ}52'57.7''$	$K^2 - 2Kr = r^2, q.p.$ $\Delta_1 = r$ $R=2414.214$ $B=34^{\circ}7'2.3''$ $C=75^{\circ}0'0''$ $c=4663.904$	$r=1000$ $r_1=4247.10$ $r_2=1831.016$ $r_3=4578.739$
20.	$a=2157.87$ $b=2132.19$ $A=64^{\circ}46'37.2''$	$R=11.92647$ $B=63^{\circ}21'58.4''$ $C=51^{\circ}51'24.4''$ $c=1875.96$	$r=586.8513$ $r_1=1955.67$ $r_2=1902.85$ $r_3=1498.918$
21.	$a=4891.32$ $b=4377.24$ $A=63^{\circ}56'43.96''$	$R=2722.309$ $B=53^{\circ}30'34.76''$ $C=62^{\circ}32'41.28''$ $c=4831.4$	$r=1347.455$ $r_1=4400.659$ $r_2=3554.227$ $r_3=4281.804$
22.	$a=10978.38$ $b=9662.8$ $A=68^{\circ}3'31.5''$	$R=5917.834$ $B=54^{\circ}43'38.54''$ $C=57^{\circ}12'49.96''$ $c=9950.22$	$r=2915.29$ $r_1=10328.493$ $r_2=7916.244$ $r_3=8341.887$
23.	$a=719.29$ $b=698.76$ $A=57^{\circ}41'11.7''$	$R=425.5465$ $B=55^{\circ}11'11.44''$ $C=67^{\circ}7'36.86''$ $c=784.17$	$r=210.2832$ $r_1=606.4242$ $r_2=575.4812$ $r_3=730.5638$
24.	$a=8625.87$ $b=7818.03$ $A=67^{\circ}37'52.48''$	$R=4663.873$ $B=56^{\circ}56'42.14''$ $C=55^{\circ}25'25.38''$ $c=7680.2$	$r=2301.68$ $r_1=8079.603$ $r_2=6541.669$ $r_3=6335.902$

$$\Delta_2 = 234.7964$$

$$\Delta_1 = 1000.0007$$

$$\Delta_2 = 603.2828$$

$$\Delta_3 = 1598.853$$

EXAMPLES VIII.—Continued.

DATA.	ANSWERS.
25. $a=11878.92$ $b=10630.44$ $A=63^{\circ}27'15.2''$	$R=6639.403$ $B=53^{\circ}10'59.98''$ $C=63^{\circ}21'44.82''$ $c=11869.4$ $r=3283.28$ $r_1=10627.62$ $r_2=8604.674$ $r_3=10608.6$
26. $a=8129.16$ $b=9246.77$ $B=63^{\circ}57'31.52''$	$R=5145.795$ $A=52^{\circ}10'28.76''$ $C=63^{\circ}51'59.72''$ $c=9239.49$ $r=2535.525$ $r_1=6515.731$ $r_2=8308.923$ $r_3=8294.053$
27. $a=8704.287$ $b=7895.319$ $A=67^{\circ}34'13.06''$	$R=4708.338$ $B=56^{\circ}58'32.84''$ $C=55^{\circ}27'14.1''$ $c=7756.236$ $r=2324.09$ $r_1=8147.836$ $r_2=6608.74$ $r_3=6400.864$
28. $a=2777.331$ $b=2028.932$ $A=81^{\circ}2'39.88''$	$R=1405.803$ $B=46^{\circ}11'20.42''$ $C=52^{\circ}45'59.7''$ $c=2238.537$ $r=636.8488$ $r_1=3010.775$ $r_2=1502.031$ $r_3=1747.254$
29. $a=6004.551$ $b=4940.314$ $A=52^{\circ}16'48.72''$	$R=3795.486$ $B=40^{\circ}36'10.9''$ $C=87^{\circ}7'0.38''$ $c=7581.363$ $r=1599.1825$ $r_1=4545.994$ $r_2=3426.808$ $r_3=8808.323$
30. $a=72027.91$ $b=63926$ $A=76^{\circ}22'16.08''$	$R=37057.4$ $B=59^{\circ}36'5.30''$ $C=44^{\circ}1'38.62''$ $c=51509.95$ $r=17070.53$ $r_1=73721.52$ $r_2=53682.39$ $r_3=37896.22$
31. $a=10301.99$ $b=9205.4$ $A=61^{\circ}16'11.5''$	$R=5874.135$ $B=51^{\circ}35'13.32''$ $C=67^{\circ}8'35.18''$ $c=10825.782$ $r=2880.918$ $r_1=8081.956$ $r_2=7329.696$ $r_3=10065.804$
32. $a=72027.91$ $b=58443.05$ $A=86^{\circ}18'43.14''$	$R=36088.7$ $B=54^{\circ}4'4.86''$ $C=39^{\circ}37'12''$ $c=46027$ $r=15209.16$ $r_1=82743.85$ $r_2=45031.10$ $r_3=31788.98$
33. $a=12785.2$ $b=11688.61$ $A=69^{\circ}4'52.18''$	$R=6843.687$ $B=58^{\circ}38'46.20''$ $C=52^{\circ}16'21.62''$ $c=10825.782$ $r=3348.419$ $r_1=12148.782$ $r_2=9913.952$ $r_3=8660.432$

IX.—Triangles in which two sides and the angle opposite the lesser are given. Ambiguous Case.

Let the two given lengths be a, b ($a > b$), and B the given angle. First calculate R by the equation $2R = b/\sin B$; then $\sin A$ may be found from $\sin A = a/2R$, but either the acute or the obtuse angle having this sine may be taken, and there are two triangles having the assigned data in all cases except when $A = 90^{\circ}$. Thus there will be two values of A , supplementary to each other. We shall denote the acute angle by A_1 , the obtuse by A_2 . The equation $A + B + C = 180^{\circ}$ supplies two corresponding values of C , which we denote by C_1, C_2 . The work is otherwise exactly the same as in VIII. The equation $b^2 = a^2 + c^2 - 2ac \cos B$ shows us that, if c_1, c_2 be the two values of c , $2a + c_1 = c_2 \cos B$, $c_1 c_2 = a^2 - b^2 \equiv (a - b)(a + b)$, and either of these may be used

to check the work. Since the angle B is common to both triangles, this angle should be used in calculating r , r_1 , r_2 , r_3 for the two triangles.

EXAMPLE WORKED OUT.

DATA.	$\log b=4.1673297$	$\log a=4.2273377$	2760(20.44"
$a=16878.65$	$-\log \sin B=1.9375306$	$-\log 2R=4.2297991$	$A_1=83^{\circ}54'20.44"$
$b=14700.42$	$\log 2R=4.2297991$	$L \sin A=9.9975386$	$C_1=36^{\circ}5'39.56"$
$B=60^{\circ}$	842	$83^{\circ}54' + 1$	$A_2=96^{\circ}5'39.56"$
	<u>16974.58</u>	135	$C_2=23^{\circ}54'20.44"$
	149	40	
	$\log 2R=4.2297991$	1733	$\log 2R=4.2297991$
	$\log \sin C_1=1.7700808$	36-1039 8	2850
	1143	3.6-104	971
	$\log r_1=4.0000002$	-.04- 1 2	20-950
			.4= 19
			.04= 2
	<u>10000.005</u>		01
		6878.645	29

$$\begin{aligned} \log (a-b) &= 3.81038 \\ \log (a+b) &= 4.4993994 \\ \log (a^2 - b^2) &= 7.8375032 \\ \log r_1 c_3 &= 7.8375032 \end{aligned}$$

$$\begin{aligned} \log \tan \frac{B}{2} &= 1.7614394 \\ \log \cot \frac{1}{2} B &= 2.385606 \end{aligned}$$

$a = 16878.65$	$s_1 - a = 3910.89$	$\log = 3.5922756$		
$b = 14700.42$	$s_1 - b = 6089.12$	$\log = 3.7845545$		
$c = 10000.01$	$s_1 - c = 10789.53$	$\log = 4.0330024$		
	$s_1 = 20789.54$	$\log = 4.3178449$		
$2s_1 = 41579.08$				
$s_1 = 20789.54$				
I.	$\log r$	$\log r_1$	$\log r_2$	$\log r_3$
	3.7845545	4.0330024	4.3178449	3.5922756
	1.7614394	.2385606	1.7614394	.2385606
	3.5459939	4.2715630	4.0792843	3.8308362
	871	28	536	24
	3515.555	68	12002.85	307
		18088.01	2	6773.86
				38
				4R = 33949.165

$a = 16878.65$	$s_2 - a = 2350.21$	$\log = 3.3711067$		
$b = 14700.42$	$s_2 - b = 4528.44$	$\log = 3.6559486$		
$c = 6878.65$	$s_2 - c = 12350.21$	$\log = 4.0916743$		
	$s_2 = 19228.86$	$\log = 4.2839535$		
$2s_2 = 38457.72$				
$s_2 = 19228.86$				
II.	$\log r$	$\log r_1$	$\log r_2$	$\log r_3$
	3.6559486	4.0916743	4.2839535	3.3711067
	1.7614394	.2385606	1.7614394	.2385606
	3.4173880	4.3302349	4.0453929	3.6096673
	720	11	621	584
	2614.497	160	11101.79	308
		21391.19	38	4070.683
				89
				4R = 33949.166

ANSWERS (I.).

$$\begin{aligned} R &= 8487.29 & r &= 3515.555 \\ A &= 83^\circ 54' 20.44'' & r_1 &= 18688.01 \\ C &= 36^\circ 5' 39.56'' & r_2 &= 12002.85 \\ c &= 10000.005 & r_3 &= 6773.86 \end{aligned}$$

ANSWERS (II.).

$$\begin{aligned} R &= 8487.29 & r &= 2614.497 \\ A &= 96^\circ 5' 39.56'' & r_1 &= 21391.19 \\ C &= 23^\circ 54' 20.44'' & r_2 &= 11101.79 \\ c &= 6878.645 & r_3 &= 4070.683 \end{aligned}$$

EXAMPLES IX.

DATA.

ANSWERS (1).

ANSWERS (2).

1. $a=3830.683$ $b=3396.549$ $B=62^{\circ}27'27.4''$	$R=1015.342$ $A=90^{\circ}$ $C=27^{\circ}32'32.6''$ $c=1771.326$	$r=668.596$ $r_1=4499.279$ $r_2=2727.953$ $r_3=1102.73$	Same as (1)	Same as (1)
2. $a=270.5115$ $b=116.5808$ $B=25^{\circ}31'43.6''$	$R=135.2558$ $A=90^{\circ}$ $C=64^{\circ}28'16.4''$ $c=244.1011$	$r=45.0852$ $r_1=315.5967$ $r_2=71.4956$ $r_3=199.0159$	Same as (1)	Same as (1)
3. $a=3650.401$ $b=2107.56$ $B=30^{\circ}$	$R=2107.56$ $A=60^{\circ}$ $C=90^{\circ}$ $c=4215.12$	$r=771.4205$ $r_1=2878.9805$ $r_2=1336.1395$ $r_3=4986.5405$	$R=2107.56$ $A=120^{\circ}$ $C=30^{\circ}$ $c=2107.56$	$r=489.0611$ $r_1=6811.7405$ $r_2=1053.78$ $r_3=1053.78$
4. $a=24150.16$ $b=22202.21$ $B=50^{\circ}59'37.5''$	$R=14285.71$ $A=57^{\circ}41'57.5''$ $C=71^{\circ}18'25''$ $c=27064.26$	$r=6918.088$ $r_1=20222.02$ $r_2=17506.51$ $r_3=26332.42$	$R=14285.71$ $A=122^{\circ}18'2.5''$ $C=6^{\circ}42'20''$ $c=3336.198$	$r=1260.025$ $r_1=45099.07$ $r_2=11848.4$ $r_3=1455.472$
5. $a=10316.36$ $b=6685.65$ $B=39^{\circ}53'33''$	$R=5212.165$ $A=8^{\circ}44'47.09''$ $C=58^{\circ}21'39.91''$ $c=887.497$	$r=2269.206$ $r_1=11197.104$ $r_2=4695.632$ $r_3=7225.329$	$R=5212.165$ $A=98^{\circ}15'12.91''$ $C=41^{\circ}51'14.09''$ $c=6955.463$	$r=1020.904$ $r_1=13841.724$ $r_2=4347.183$ $r_3=4580.709$
6. $a=10316.36$ $b=6683.635$ $B=39^{\circ}53'33''$	$R=5210.6665$ $A=8^{\circ}51'55.83''$ $C=58^{\circ}14'31.17''$ $c=8860.933$	$r=2267.026$ $r_1=11213.669$ $r_2=4692.57$ $r_3=7203.212$	$R=5210.6665$ $A=86^{\circ}8'4.17''$ $C=41^{\circ}58'22.83''$ $c=6969.593$	$r=1023.817$ $r_1=13819.64$ $r_2=4349.64$ $r_3=4597.275$
7. $a=346.4537$ $b=301.3984$ $B=60^{\circ}153.4''$	$R=173.95725$ $A=8^{\circ}44'52.3''$ $C=35^{\circ}13'14.3''$ $c=200.6515$	$r=70.97449$ $r_1=387.0413$ $r_2=245.09745$ $r_3=134.6647$	$R=173.95725$ $A=95^{\circ}15'7.7''$ $C=24^{\circ}42'58.9''$ $c=145.3837$	$r=55.00991$ $r_1=434.8742$ $r_2=229.1328$ $r_3=86.8318$
8. $a=3156.045$ $b=3013.734$ $B=72^{\circ}13'57.6''$	$R=1582.339$ $A=85^{\circ}46'1.3''$ $C=22^{\circ}01.1''$ $c=1185.525$	$r=484.4273$ $r_1=3415.513$ $r_2=2683.396$ $r_3=714.8735$	$R=1582.339$ $A=94^{\circ}13'58.7''$ $C=13^{\circ}32'3.7''$ $c=740.625$	$r=322.116$ $r_1=3720.385$ $r_2=2521.085$ $r_3=410.002$

EXAMPLES IX. — Continued.

DATA.	ANSWERS (1).			ANSWERS (2).		
	Δ	Δ_1	Δ_2	Δ	Δ_1	Δ_2
9. $a=7631.229$ $b=4973.381$ $B=39^\circ 53' 46.45''$	$A=38^\circ 6' 9.8''$ $A=79^\circ 47' 32.86''$ $C=60^\circ 18' 40.69''$ $c=6736.086$	$r=1704.741$ $r_1=8684.589$ $r_2=3509.805$ $r_3=5618.257$	$r=1704.741$ $r_1=8684.589$ $r_2=3509.805$ $r_3=5618.257$	$A=38^\circ 6' 9.8''$ $A=100^\circ 12' 27.14''$ $C=39^\circ 53' 46.41''$ $c=4973.38$	$r=1384.838$ $r_1=10512.927$ $r_2=3189.923$ $r_3=3189.920$	$r=1384.838$ $r_1=10512.927$ $r_2=3189.923$ $r_3=3189.920$
10. $a=4731.234$ $b=2718.108$ $B=32^\circ 3' 56''$	$A=2560.001$ $A=67^\circ 31' 40.31''$ $C=72^\circ 14' 20.09''$ $c=5048.411$	$r=1014.619$ $r_1=4177.595$ $r_2=1795.749$ $r_3=5281.324$	$r=1014.619$ $r_1=4177.595$ $r_2=1795.749$ $r_3=5281.324$	$A=2560.011$ $A=112^\circ 28' 19.69''$ $C=35^\circ 27' 42.71''$ $c=2970.436$	$r=716.047$ $r_1=7793.115$ $r_2=1497.173$ $r_3=1665.803$	$r=716.047$ $r_1=7793.115$ $r_2=1497.173$ $r_3=1665.803$
11. $a=481.2793$ $b=360.8528$ $B=46^\circ 44' 10.4''$	$A=247.7683$ $A=76^\circ 13' 21.66''$ $C=57^\circ 27' 9.4''$ $c=415.7854$	$r=115.8427$ $r_1=493.3678$ $r_2=271.7591$ $r_3=341.7890$	$r=115.8427$ $r_1=493.3678$ $r_2=271.7591$ $r_3=341.7890$	$A=247.7683$ $A=103^\circ 46' 38.34''$ $C=29^\circ 29' 11.26''$ $c=243.9119$	$r=78.7113$ $r_1=692.2594$ $r_2=234.6279$ $r_3=142.8972$	$r=78.7113$ $r_1=692.2594$ $r_2=234.6279$ $r_3=142.8972$
12. $a=59406.24$ $b=22733.78$ $B=22^\circ 30'$	$A=29703.12$ $A=90^\circ$ $C=67^\circ 30'$ $c=54884.2$	$r=9105.87$ $r_1=68512.11$ $r_2=13627.91$ $r_3=45778.33$	$r=9105.87$ $r_1=68512.11$ $r_2=13627.91$ $r_3=45778.33$	Same as (1)	Same as (1)	Same as (1)
13. $a=964701$ $b=955078$ $B=59^\circ 59' 23.7''$	(1) $\Delta_1=9628.197$ (2) $\Delta_1=542988.5$	$r=551470.5$ $A=61^\circ 0' 18.17''$ $C=59^\circ 0' 18.13''$ $c=915454.9$	$r=275651.2$ $r_1=843060.8$ $r_2=850953.5$ $r_3=810618.8$	$A=551470.5$ $A=118^\circ 59' 41.83''$ $C=1^\circ 54' 47''$ $c=19539.84$	$r=8416.88$ $r_1=1645989.8$ $r_2=559719.5$ $r_3=8589.98$	$r=8416.88$ $r_1=1645989.8$ $r_2=559719.5$ $r_3=8589.98$
14. $a=337.573$ $b=294.0084$ $B=60^\circ$	(1) $\Delta_1=70.311$ $\Delta_2=70.311$ (2) $\Delta_1=105.17894$	$A=169.74584$ $A=83^\circ 54' 20.44''$ $C=36^\circ 5' 39.56''$ $c=200$	$r=70.31104$ $r_1=373.76052$ $r_2=240.05694$ $r_3=135.47704$	$A=169.74584$ $A=96^\circ 5' 39.56''$ $C=23^\circ 54' 20.44''$ $c=137.57292$	$r=52.28994$ $r_1=427.82382$ $r_2=222.03568$ $r_3=81.41366$	$r=52.28994$ $r_1=427.82382$ $r_2=222.03568$ $r_3=81.41366$
15. $a=6142.887$ $b=6127.56$ $B=59^\circ 45' 12''$	(1) $\Delta_1=30.539$ (2) $\Delta_2=6158.099$	$A=3546.597$ $A=60^\circ$ $C=60^\circ 14' 48''$ $c=6158.099$	$r=1773.267$ $r_1=5319.862$ $r_2=5293.450$ $r_3=5346.341$	$A=3546.597$ $A=120^\circ$ $C=0^\circ 14' 48''$ $c=30.539$	$r=13.17464$ $r_1=10652.965$ $r_2=3533.357$ $r_3=13.23971$	$r=13.17464$ $r_1=10652.965$ $r_2=3533.357$ $r_3=13.23971$
16. $a=775.2704$ $b=674.2277$ $B=60^\circ$	(1) $\Delta_1=A_1=166.874$ $\Delta_2=325.9911$ (2) $\Delta_1=A_1=35.7022$ $\Delta_2=149.2793$	$A=389.2655$ $A=81^\circ 45' 15''$ $C=35^\circ 14' 45''$ $c=449.2793$	$r=158.8643$ $r_1=866.2148$ $r_2=548.13$ $r_3=301.5821$	$A=389.2655$ $A=95^\circ 14' 45''$ $C=24^\circ 45' 15''$ $c=325.9911$	$r=123.2741$ $r_1=972.0854$ $r_2=512.5396$ $r_3=194.8110$	$r=123.2741$ $r_1=972.0854$ $r_2=512.5396$ $r_3=194.8110$

EXAMPLES IX. —Continued.

DATA.	ANSWERS (1).		ANSWERS (2).	
17. $a=701.37$ $b=666.3$ $B=54^{\circ} 9.25''$	$A=411.7826$ $A=58^{\circ} 23.20.5''$ $C=67^{\circ} 36.30.25''$ $c=761.47$	$r=202.9399$ $r_1=594.8341$ $r_2=542.4556$ $r_3=712.7807$	$\lambda=411.7826$ $A=121^{\circ} 36.39.5''$ $C=4^{\circ} 23.11.25''$ $c=629.89$	$r=24.9832$ $r_1=1280.219$ $r_2=364.4987$ $r_3=27.3955$
18. $a=393.75$ $b=391.54$ $B=46^{\circ} 15.35''$	$A=216.4135$ $A=65^{\circ} 27.55.58''$ $C=49^{\circ} 45.46.06''$ $c=330.41$	$r=125.4844$ $r_1=358.589$ $r_2=353.8239$ $r_3=238.7252$	$R=216.4135$ $A=114^{\circ} 32.1.42''$ $C=0^{\circ} 41.43.22''$ $c=3.25254$	$r=2.36661$ $r_1=614.9157$ $r_2=250.7062$ $r_3=2.39848$
19. $a=5265.68$ $b=5222.3$ $B=65^{\circ} 28.86''$	$A=2650.415$ $A=83^{\circ} 23.53.3''$ $C=31^{\circ} 8.3.84''$ $c=2740.78$	$r=1023.421$ $r_1=5714.817$ $r_2=4123.293$ $r_3=1786.97$	$R=2650.415$ $A=96^{\circ} 36.6.7''$ $C=17^{\circ} 55.50.44''$ $c=1631.947$	$r=667.0308$ $r_1=5577.292$ $r_2=376.902$ $r_3=924.4924$
20. $a=688.9$ $b=627.77$ $B=64^{\circ} 16.94''$	$A=348.4346$ $A=81^{\circ} 19.36.02''$ $C=34^{\circ} 24.15.04''$ $c=393.75$	$r=142.8642$ $r_1=734.6451$ $r_2=537.1917$ $r_3=264.7658$	$R=348.4346$ $A=98^{\circ} 40.23.98''$ $C=17^{\circ} 32.7.05''$ $c=204.414$	$r=83.40002$ $r_1=885.3568$ $r_2=477.7268$ $r_3=114.0542$
21. $a=313.885$ $b=165.205$ $B=35^{\circ} 44.5.06''$	$A=190.2383$ $A=55^{\circ} 35.10.02''$ $C=98^{\circ} 40.44.32''$ $c=376.12$	$r=59.94535$ $r_1=225.3852$ $r_2=97.67535$ $r_3=497.8303$	$R=190.2383$ $A=124^{\circ} 24.49.38''$ $C=29^{\circ} 51.5.56''$ $c=189.384$	$r=38.61218$ $r_1=634.1209$ $r_2=76.35015$ $r_3=89.09454$
22. $a=10978.38$ $b=9662.8$ $B=54^{\circ} 43.38.54''$	$A=5917.834$ $A=86^{\circ} 33.31.5''$ $C=57^{\circ} 12.49.96''$ $c=9950.22$	$r=2915.29$ $r_1=10328.493$ $r_2=7916.244$ $r_3=8341.887$	$\lambda=5917.834$ $A=111^{\circ} 56.28.5''$ $C=13^{\circ} 19.52.96''$ $c=2729.1$	$r=1046.656$ $r_1=17304.79$ $r_2=6047.607$ $r_3=1365.556$
23. $a=1908.237$ $b=633.911$ $B=39^{\circ} 20.21.35''$	$A=500$ $A=86^{\circ} 35.50.4''$ $C=54^{\circ} 3.48.25''$ $c=809.6069$	$r=209.8239$ $r_1=1150.473$ $r_2=436.4176$ $r_3=622.9357$	$\lambda=500$ $A=93^{\circ} 24.9.6''$ $C=47^{\circ} 15.29.05''$ $c=734.4185$	$r=106.375$ $r_1=1255.73$ $r_2=422.9687$ $r_3=517.6794$
24. $a=4457.15$ $b=4164.894$ $B=54^{\circ} 11.18.24''$	$A=2573.337$ $A=60^{\circ}$ $C=65^{\circ} 58.41.76''$ $c=4700.928$	$r=1272.674$ $r_1=3846.01$ $r_2=3395.787$ $r_3=1324.224$	$R=2573.337$ $A=120^{\circ}$ $C=5^{\circ} 58.41.76''$ $c=536.0318$	$r=211.116$ $r_1=7931.125$ $r_2=2334.23$ $r_3=239.1052$

X.—Tetrahedra, the lengths of the edges being given.

In a tetrahedron $OABC$ let the lengths of OA , OB , OC be denoted by a , b , c ; the lengths of the respectively opposite edges BC , CA , AB by x , y , z ; the dihedral angles opposite to a , b , c , x , y , z respectively by A , B , C , X , Y , Z . Also, let any angle of a triangular face which is opposite to OA or BC be denoted by α , one opposite to OB or CA by β , one opposite to OC or AB by γ ; all the angles at O having the suffix 1, those at A the suffix 2, those at B the suffix 3, and those at C the suffix 4. Then, in each of the four triangular faces we have the lengths of the three sides given, and may calculate the angles as in VI. In the face ABC opposite O the lengths are x , y , z , the angles α_2 , β_2 , γ_1 ; in the face opposite A the lengths are x , b , c , the angles α_1 , β_4 , γ_3 ; in the face opposite B the lengths are a , y , c , the angles α_4 , β_1 , γ_2 ; lastly, in the face opposite C the lengths are a , b , z , the angles α_3 , β_3 , γ_4 . The sums of the lengths are denoted by $2s_1$, $2s_2$, $2s_3$, $2s_4$. The sum of the plane angles at O is similarly $2\sigma_1$, and the sums of the angles at A , B , C are $2\sigma_2$, $2\sigma_3$, $2\sigma_4$. The volume V may be found from each of the equations

$$\begin{aligned} 3V &= abc \sqrt{\sin \sigma_1 \sin (\sigma_1 - \alpha_1)} \sqrt{\sin (\sigma_1 - \beta_1) \sin (\sigma_1 - \gamma_1)} \\ &= ayz \sqrt{\sin \sigma_2 \sin (\sigma_2 - \alpha_2)} \sqrt{\sin (\sigma_2 - \beta_2) \sin (\sigma_2 - \gamma_2)} \\ &= abx \sqrt{\sin \sigma_3 \sin (\sigma_3 - \alpha_3)} \sqrt{\sin (\sigma_3 - \beta_3) \sin (\sigma_3 - \gamma_3)} \\ &= xyz \sqrt{\sin \sigma_4 \sin (\sigma_4 - \alpha_4)} \sqrt{\sin (\sigma_4 - \beta_4) \sin (\sigma_4 - \gamma_4)}. \end{aligned}$$

The most convenient manner of arranging the work will be obvious on inspecting the one calculated. The dihedral angles at O are X , Y , Z , and these are to be determined by the equations

$$\begin{aligned} \tan \frac{1}{2} X \sin (\sigma_1 - \alpha_1) &= \tan \frac{1}{2} Y \sin (\sigma_1 - \beta_1) = \tan \frac{1}{2} Z \sin (\sigma_1 - \gamma_1) \\ &= \sqrt{\frac{\sin (\sigma_1 - \alpha_1) \sin (\sigma_1 - \beta_1) \sin (\sigma_1 - \gamma_1)}{\sin \sigma_1}}, \end{aligned}$$

no fresh logarithms needing to be taken out. This part of the work is almost exactly the same as that of calculating the tangents of the half angles of a triangle when the sides are given. The dihedral angles at A are X , B , C ; at B are A , Y , C ; and at C are A , B , Z ; and are to be found in the same way from the angles with suffixes 2, 3, 4 respectively. Thus the volume is calculated from four different expressions, and each dihedral angle from two; and the agreement of the results is sufficient to guarantee the correctness of the calculations. From each tetrahedron we get the six parts of each of four plane triangles, and the six parts of each of four spherical triangles; and these may be used to supply any deficiency in any of the preceding sets of examples; or for practice in solving spherical triangles. To make these plane and spherical triangles complete, I have given always the values of all the plane angles of each face of every tetrahedron, or rather the halves of these angles. The order of magnitude of $A+X$, $B+Y$, $C+Z$ being always the same as that of $a+x$, $b+y$, $c+z$; that of $A-X$, $B-Y$, $C-Z$, the same as that of $a-x$, $b-y$, $c-z$; that of $A-X$, $Y-B$, $Z-C$, the same as that of $a-x$, $y-b$, $z-c$, etc.; the results should be tested accordingly. When

two terms of one system of lengths are equal, the two corresponding terms of the corresponding system of angles will be also equal. In many cases, the data will be found to satisfy one, or two, of such equations, and it will be seen that the calculated angles satisfy the corresponding equations, *quam prox.* Many other properties of tetrahedra are illustrated, attention being directed to such cases by a very brief entry in column of remarks. The tetrahedron in which a, b, c are the lengths of three conterminous edges, and x, y, z the lengths of the edges respectively opposite

to a, b, c , is here denoted by $\begin{Bmatrix} a & x \\ b & y \\ c & z \end{Bmatrix}$. The same tetrahedron will thus be also

denoted by $\begin{Bmatrix} a & x \\ y & b \\ z & c \end{Bmatrix}$, $\begin{Bmatrix} x & a \\ b & y \\ c & z \end{Bmatrix}$, or $\begin{Bmatrix} x & a \\ y & b \\ c & z \end{Bmatrix}$. When six lengths are given, there will

be thirty different tetrahedra having these for edges, but some of these may be impossible, even when any set of three of the lengths are sides of a real triangle. In dealing with such a system of tetrahedra, it seems most convenient to arrange them in five groups of six, the longest edge having the same edge opposite to it in all the tetrahedra of any one group.

The sums of the dihedral angles at O, A, B, C respectively are denoted by $2S_1, 2S_2, 2S_3, 2S_4$; and the order of magnitude of S_1, S_2, S_3, S_4 corresponds to that of s_1, s_2, s_3, s_4 . When $a - x = b - y = c - z$, $s_2 = s_3 = s_4$, $S_2 = S_3 = S_4$; and, if $s_2 > s_3 > s_4$, it will be observed that

$$\sin \sigma_2 \sin (\sigma_2 - \alpha_2) \sin (\sigma_2 - \beta_2) \sin (\sigma_2 - \gamma_2) < \sin \sigma_3 \sin (\sigma_3 - \alpha_3) \sin (\sigma_3 - \beta_3) \sin (\sigma_3 - \gamma_3) \\ < \sin \sigma_4 \sin (\sigma_4 - \alpha_4) \sin (\sigma_4 - \beta_4) \sin (\sigma_4 - \gamma_4);$$

also

$$\sin (\sigma_2 - \alpha_2) \sin (\sigma_2 - \beta_2) \sin (\sigma_2 - \gamma_2) / \sin \sigma_2 < \sin (\sigma_3 - \alpha_3) \sin (\sigma_3 - \beta_3) \sin (\sigma_3 - \gamma_3) / \sin \sigma_3 \\ < \sin (\sigma_4 - \alpha_4) \sin (\sigma_4 - \beta_4) \sin (\sigma_4 - \gamma_4) / \sin \sigma_4.$$

Many cases of maximum volume under certain conditions will be found.

From the equations of which $\frac{\partial V}{\partial a} = \frac{1}{8}ax \cot A$ is the type, we see that when

a, x vary subject to the condition that $a + x$ is constant, the volume (V) will be a maximum when $A = X$; when subject to the condition that $a - x$ is constant, V will be a maximum when $A + X = 180^\circ$; and similarly with the sum or difference of any pair of opposite edges. When $a, x, b + y, c + z$ are given, V will be a maximum when $B = Y, C = Z$, which involves $b = y, c = z$. When $a, x, b + y, c - z$ are given, V will be a maximum when $B = Y, C + Z = 180^\circ$. As the lengths of the edges are varied so as to approach this maximum value, it appears that the variations of b and y do not alter $C + Z$; nor those of c and z affect $B - Y$. When $a, x, b - y, c - z$ are given, it should seem that V will be a maximum when $B + Y = C + Z = 180^\circ$, but this maximum will be when b, y, c, z are infinite, and consequently V .

In any tetrahedron in which $\alpha_1 + \beta_1 + \gamma_1 = \alpha_2 + \beta_2 + \gamma_2 = \pi$, it will be found that

$$\gamma_1 + \alpha_1 - \beta_1 = \gamma_4 + \alpha_4 - \beta_4; \quad \alpha_1 + \beta_1 - \gamma_1 = \alpha_3 + \beta_3 - \gamma_3; \\ \gamma_2 + \alpha_2 - \beta_2 = \gamma_3 + \alpha_3 - \beta_3; \quad \alpha_2 + \beta_2 - \gamma_2 = \alpha_4 + \beta_4 - \gamma_4;$$

$$\begin{cases} a=3.5683, & x=4.9792 \\ b=3.6767, & y=5.0876 \\ c=4.2235, & z=5.6344 \end{cases}$$

Calculation of the Angles of the Faces.

$x=4.9792$	$s_1-x=2.8714$	$\log(s_1-x)=.4580937$	10.1750904	$57960(18.81)$	10.1750904	$90360(29.95)$	10.1750904	$30540(11.21)$	$\frac{1}{2}a=27^\circ 31' 41.18''$
$y=5.0876$	$s_1-y=2.7630$	$\log(s_1-y)=.4413809$	$.4580937$	27140	$.4413809$	30020	$.3450090$	3300	$\frac{1}{2}b=28^\circ 26' 30.04''$
$z=5.6344$	$s_1-z=2.2162$	$\log(s_1-z)=.3456090$	2484	18	9.7160967	2867	9.8201814	576	$\frac{1}{2}c=34^\circ 1' 48.78''$
15.7012	$s_1=7.8506$	$-\log s_1=-.8949028$	$21^\circ 32' - 1^\circ 09.33$	3082	$25^\circ 27' - 1^\circ 80.01$	152	$34^\circ 2' - 1^\circ 53.23$	31	$90^\circ 0' 0''$
$s_1=7.8506$		$2) 3501808$	3082	966	3017	1506	2724	509	
$x=4.9792$	$s_2-x=1.4605$	$\log(s_2-x)=.1645016$	10.0713130	$4440(1.72)$	10.0713130	$1422(4.56)$	10.0713130	$17760(5.83)$	$\frac{1}{2}a=38^\circ 53' 58.26''$
$y=3.6767$	$s_2-y=2.7630$	$\log(s_2-y)=.4413809$	$.1645016$	1855	$.4413809$	22	$.3450090$	2525	$\frac{1}{2}b=23^\circ 5' 55.93''$
$z=4.2235$	$s_2-z=2.2162$	$\log(s_2-z)=.3456090$	9.9068114	45	9.6299321		9.7257040	87	$\frac{1}{2}c=28^\circ 0' 5.81''$
12.8794	$s_2=6.4397$	$-\log s_2=-.8688656$	$35^\circ 34' - 2^\circ 58.5$	88	$23^\circ 6' - 1^\circ 558$		$28^\circ 0' + 1^\circ 67.44$		$90^\circ 0' 0''$
$s_2=6.4397$		$2) 1426259$	2585	74	3500	237	3047	296	
$x=3.5683$	$s_3-x=2.8714$	$\log(s_3-x)=.4580937$	10.0629230	$74160(20.34)$	10.0629230	$57780(22.59)$	10.0629230	$52440(17.02)$	$\frac{1}{2}a=21^\circ 55' 39.61''$
$y=5.0876$	$s_3-y=1.3521$	$\log(s_3-y)=.1310088$	4580937	1220	$.1310088$	6620	$.3450090$	2163	$\frac{1}{2}b=40^\circ 31' 37.40''$
$z=4.2235$	$s_3-z=2.2162$	$\log(s_3-z)=.3456090$	9.6048293	126	9.9319142	1504	9.7173140	63	$\frac{1}{2}c=27^\circ 32' 42.96''$
12.8794	$s_3=6.4397$	$-\log s_3=-.8688656$	$21^\circ 56' - 1^\circ 93.29$	9329	$40^\circ 32' - 1^\circ 201.05$	225	$27^\circ 33' - 1^\circ 401.4$		$50^\circ 0' 0''$
$s_3=6.4397$		1258459	3647	1236	2558	963	3081	874	
$x=3.5683$	$s_4-x=2.8714$	$\log(s_4-x)=.4580937$	9.9982834	$33360(8.17)$	9.9982834	$106200(26.82)$	9.9982834	$48120(18.62)$	$\frac{1}{2}a=19^\circ 7' 51.82''$
$y=3.6767$	$s_4-y=2.7630$	$\log(s_4-y)=.4413809$	$.4580937$	704	$.4413809$	2702	1.9059577	22280	$\frac{1}{2}b=19^\circ 49' 56.81''$
$z=5.6344$	$s_4-z=2.8053$	$\log(s_4-z)=.1901057$	9.5401807		9.5569025	3566	10.0923257	1608	$\frac{1}{2}c=51^\circ 2' 41.37''$
12.8794	$s_4=6.4397$	$-\log s_4=-.8688656$	$19^\circ 8' - 1^\circ 24.53$	2453	$19^\circ 49' + 1^\circ 72.55$	99	$51^\circ 31' + 1^\circ 40.59$	58	$90^\circ 0' 0''$
$s_4=6.4397$		1.9965667	4082	556	3959	1770	2584	802	

Calculation of the Volume.

$\alpha_1 = 77^\circ 47' 56.52''$ $\beta_1 = 81^\circ 3' 14.80''$ $\gamma_1 = 102^\circ 5' 22.74''$	$\alpha_2 = 55^\circ 3' 22.36''$ $\beta_2 = 39^\circ 38' 53.62''$ $\gamma_2 = 55^\circ 5' 25.92''$	$341 : 6 - \frac{1}{2} + \frac{1}{10}$ 305 7 -17 +8	$\alpha_3 = 38^\circ 15' 43.61''$ $\beta_3 = 56^\circ 53' 0.08''$ $\gamma_3 = 56^\circ 0' 11.62''$	$325 : 53 + \frac{1}{2}$ 138 4 9 2 2 3	$\alpha_4 = 43^\circ 51' 19.28''$ $\beta_4 = 46^\circ 11' 51.86''$ $\gamma_4 = 68^\circ 3' 37.56''$	$244 : 4 \frac{1}{2}$ 97 4 1 6
$260^\circ 56' 34.06''$ $\sigma_1 = 130^\circ 28' 17.03''$	$149^\circ 47' 41.90''$ $\sigma_2 = 74^\circ 53' 50.95''$	$3501 : 30 = 1150$ -1.5 -87 5 + .09= 5 3	$151^\circ 8' 55.34''$ $\sigma_3 = 75^\circ 34' 27.67''$	$1658 : 40 = 1105$ 4= 110 8 .03=	$158^\circ 6' 48.70''$ $\sigma_4 = 79^\circ 3' 24.35''$	$1791 : 5 = 149$ 3 .06= 1 8 01= 3
$\sigma_1 - \alpha_1 = 52^\circ 40' 20.51''$ $\sigma_1 - \beta_1 = 49^\circ 25' 2.23''$ $\sigma_1 - \gamma_1 = 28^\circ 22' 54.29''$	$\sigma_2 - \alpha_2 = 19^\circ 50' 28.59''$ $\sigma_2 - \beta_2 = 35^\circ 14' 57.33''$ $\sigma_2 - \gamma_2 = 19^\circ 48' 25.03''$	$1788 : 54 = 1609$ 3= 89 4 3= 8 9 .03= 9	$\sigma_3 - \alpha_3 = 37^\circ 18' 44.03''$ $\sigma_3 - \beta_3 = 18^\circ 41' 27.59''$ $\sigma_3 - \gamma_3 = 19^\circ 34' 16.05''$	$3738 : 20 = 1246$ 7.5= 467 3 .09= 5 6 5 6	$\sigma_4 - \alpha_4 = 35^\circ 12' 5.07''$ $\sigma_4 - \beta_4 = 32^\circ 51' 32.40''$ $\sigma_4 - \gamma_4 = 10^\circ 59' 46.79''$	$1956 : 30 = 978$ 2.4= 78 2 .09= 2 9
$\sigma_1 = 130^\circ 28' 17.03''$ $86^\circ - \sigma_1 = 49^\circ 31' 42.97''$	$\sigma_2 = 74^\circ 53' 50.95''$	$3508 : 20 = 1169$ 5= 292 3 .03= 1 8	$\sigma_3 = 75^\circ 34' 27.67''$	$3553 : 15 = 888$ 1= 59 2 .03= 3	$\sigma_4 = 79^\circ 3' 24.35''$	$6504 : 48 = 5203$ -1.2= 130 1 -01= -1 1
$L \sin \sigma_1 = 9.8811534$ $L \sin (\sigma_1 - \alpha_1) = 9.0004331$ $L \sin (\sigma_1 - \beta_1) = 9.8805952$ $L \sin (\sigma_1 - \gamma_1) = 9.6767963$	$L \sin \sigma_2 = 9.9847059$ $L \sin (\sigma_2 - \alpha_2) = 9.5305650$ $L \sin (\sigma_2 - \beta_2) = 9.7611063$ $L \sin (\sigma_2 - \gamma_2) = 9.5298638$	289 5 1668 3 1708 4 1463 4 5130 2) 2.8067540	$L \sin \sigma_3 = 9.0860720$ $L \sin (\sigma_3 - \alpha_3) = 9.7824643$ $L \sin (\sigma_3 - \beta_3) = 9.5036077$ $L \sin (\sigma_3 - \gamma_3) = 9.5249196$	149 9 1216 6 1718 9 930 5 4036 2) 2.7994672	$L \sin \sigma_4 = 9.9920201$ $L \sin (\sigma_4 - \alpha_4) = 9.7607483$ $L \sin (\sigma_4 - \beta_4) = 9.7343529$ $L \sin (\sigma_4 - \gamma_4) = 9.2799484$	99 0 151 4 1059 1 5072 0 6381 2) 2.7677078
1.6696669 $\log \sigma = .5324614$ $\log \beta = .5054582$ $\log \sigma = .6256725$ $-\log 3 = .4771213$ $\log V = .9360777$	1.4033770 $\log \sigma = .5524014$ $\log \beta = .7065130$ $\log \sigma = .7508477$ $-\log 3 = .4771213$ $\log V = .9360778$	1.4033770 $\log \sigma = .5524014$ $\log \beta = .7065130$ $\log \sigma = .7508477$ $-\log 3 = .4771213$ $\log V = .9360778$	1.3997336 $\log \sigma = .6971596$ $\log \beta = .5654582$ $\log \sigma = .7508477$ $-\log 3 = .4771213$ $\log V = .9360778$	1.3838539 $\log \sigma = .6971596$ $\log \beta = .7065130$ $\log \sigma = .6256725$ $-\log 3 = .4771213$ $\log V = .9360777$	1.3838539 $\log \sigma = .6971596$ $\log \beta = .7065130$ $\log \sigma = .6256725$ $-\log 3 = .4771213$ $\log V = .9360777$	1.3838539 $\log \sigma = .6971596$ $\log \beta = .7065130$ $\log \sigma = .6256725$ $-\log 3 = .4771213$ $\log V = .9360777$
1.6696669 $\log \sigma = .5324614$ $\log \beta = .5054582$ $\log \sigma = .6256725$ $-\log 3 = .4771213$ $\log V = .9360777$	1.4033770 $\log \sigma = .5524014$ $\log \beta = .7065130$ $\log \sigma = .7508477$ $-\log 3 = .4771213$ $\log V = .9360778$	1.4033770 $\log \sigma = .5524014$ $\log \beta = .7065130$ $\log \sigma = .7508477$ $-\log 3 = .4771213$ $\log V = .9360778$	1.3997336 $\log \sigma = .6971596$ $\log \beta = .5654582$ $\log \sigma = .7508477$ $-\log 3 = .4771213$ $\log V = .9360778$	1.3838539 $\log \sigma = .6971596$ $\log \beta = .7065130$ $\log \sigma = .6256725$ $-\log 3 = .4771213$ $\log V = .9360777$	1.3838539 $\log \sigma = .6971596$ $\log \beta = .7065130$ $\log \sigma = .6256725$ $-\log 3 = .4771213$ $\log V = .9360777$	1.3838539 $\log \sigma = .6971596$ $\log \beta = .7065130$ $\log \sigma = .6256725$ $-\log 3 = .4771213$ $\log V = .9360777$

Calculation of the Dihedral Angles.

$-L \sin \sigma_1 = -9.8812366$	8	19.7883763	19.7883763	19.7883763	-9.9847348	5	19.4186421	19.4186421
$L \sin(\sigma_1 - \alpha_1) = 9.9004660$	2	9.9004660	9.8805092	9.6770079	9.5307318	3	9.5307318	9.5300101
$L \sin(\sigma_1 - \beta_1) = 9.8805092$	2				9.7612771	4		
$L \sin(\sigma_1 - \gamma_1) = 9.6770079$	4	$L \tan \frac{1}{2} X$	$L \tan \frac{1}{2} Y$	$L \tan \frac{1}{2} Z$	9.5300101	4	$L \tan \frac{1}{2} Y$	$L \tan \frac{1}{2} C$
2) 19.5767525		9.8879103	9.9078671	10.1113684	2) 18.8372842	6	9.8879103	9.8886320
$-L \sin \sigma_2 = -9.9860869$	9	19.4136466	19.4136466	19.4136466	-9.9920300		19.3918239	19.3918239
$L \sin(\sigma_2 - \alpha_2) = 9.7825859$	6	9.7825860	9.5057796	9.5250146	9.7607634	4	9.7607634	9.7344588
$L \sin(\sigma_2 - \beta_2) = 9.9057795$	9				9.7344588	1		9.2804556
$L \sin(\sigma_2 - \gamma_2) = 9.5250146$	5	$L \tan \frac{1}{2} X$	$L \tan \frac{1}{2} Y$	$L \tan \frac{1}{2} C$	9.2804556		$L \tan \frac{1}{2} X$	$L \tan \frac{1}{2} Z$
2) 18.8272932	1	9.6310666	9.9078670	9.8886320	18.7836478	5	9.6310665	10.1113683
$L \tan \frac{1}{2} A = 9.6310666$		33.40(9.52	$L \tan \frac{1}{2} B = 9.6573651$	900(27	$L \tan \frac{1}{2} C = 9.8886320$		3960(1.52	
23' 9" +	032	1803	24' 26" +	36	37' 44" -		86	
3493 }	554	56	3353 }	15	2611 }		43	
$L \tan \frac{1}{2} X = 9.8879103$		32040(12.62	$L \tan \frac{1}{2} Y = 9.9078671$	8760(3.39	$L \tan \frac{1}{2} Z = 10.1113684$		4200(1.61	
37' 41" +	8554	6830	38' 58" +	525	52' 16" +		14	
2611 }	549	1608	2584 }	146	2611 }		70	

ANSWERS.

$\frac{1}{2} A = 23^\circ 9' 9.50''$	$\frac{1}{2} B = 24^\circ 26' 0.25''$	$\frac{1}{2} C = 37^\circ 43' 58.48''$	$\sigma_1 = 130^\circ 28' 17.03''$
$\frac{1}{2} X = 37^\circ 41' 12.63''$	$\frac{1}{2} Y = 38^\circ 58' 3.38''$	$\frac{1}{2} Z = 52^\circ 16' 1.61''$	$\sigma_2 = 74^\circ 53' 50.95''$
$\frac{1}{2} (A + X) = 60^\circ 50' 22.13''$	$\frac{1}{2} (B + Y) = 63^\circ 24' 3.63''$	$\frac{1}{2} (Z + C) = 90^\circ 0' 0.09''$	$\sigma_3 = 75^\circ 34' 27.67''$
$\frac{1}{2} (X - A) = 14^\circ 32' 3.13''$	$\frac{1}{2} (Y - B) = 14^\circ 32' 3.13''$	$\frac{1}{2} (Z - C) = 14^\circ 32' 3.13''$	$\sigma_4 = 79^\circ 3' 24.35''$
$a + x = 8.5475$	$b + y = 8.7643$	$z + z = 9.8579$	$360^\circ 0' 0''$
$x - a = 1.4109$	$y - b = 1.4109$	$z - c = 1.4109$	

EXAMPLES. X.

DATA.	$\frac{1}{2}a_1, \frac{1}{2}b_1, \frac{1}{2}c_1$	$\frac{1}{2}a_2, \frac{1}{2}b_2, \frac{1}{2}c_2$	$\frac{1}{2}a_3, \frac{1}{2}b_3, \frac{1}{2}c_3$	$\frac{1}{2}a_4, \frac{1}{2}b_4, \frac{1}{2}c_4$	$\frac{1}{2}A, \frac{1}{2}B, \frac{1}{2}C$	$\frac{1}{2}A, \frac{1}{2}B, \frac{1}{2}C$	VOLUME.	REMARKS.
1. $\begin{cases} a=5.4669, \\ b=5.6185, \\ c=5.6661, \end{cases}$	$\begin{cases} 43.37^{\circ}59.08', \\ 47.33^{\circ}7.24', \\ 47.34^{\circ}34.13' \end{cases}$	$\begin{cases} 28^{\circ}16', 2.20', \\ 21^{\circ}34'11.80', \\ 21^{\circ}33'12.17' \end{cases}$	$\begin{cases} 20^{\circ}51'14.07', \\ 30^{\circ}49'33.47', \\ 23^{\circ}9'0.95' \end{cases}$	$\begin{cases} 20^{\circ}53'40.59', \\ 32^{\circ}12'59.97', \\ 30^{\circ}54'24.33' \end{cases}$	$\begin{cases} 24^{\circ}26'10.46', \\ 27^{\circ}29'34.59', \\ 27^{\circ}28'3.80' \end{cases}$	$\begin{cases} 43^{\circ}51'10.86'', \\ 47^{\circ}28'29.40'', \\ 47^{\circ}28'0.18'' \end{cases}$	28.44716	$S_1=\sigma_1=138^{\circ}45'40.44''$
2. $\begin{cases} a=7.4779, \\ b=8.1235, \\ c=8.4367, \end{cases}$	$\begin{cases} 10^{\circ}15'31.94', \\ 17^{\circ}18'32.99', \\ 16^{\circ}35'47.81' \end{cases}$	$\begin{cases} 33^{\circ}33'48.79', \\ 40^{\circ}39'13.97', \\ 41^{\circ}55'14.18' \end{cases}$	$\begin{cases} 30^{\circ}46'12.83', \\ 33^{\circ}43'28.50', \\ 37^{\circ}0'59.50' \end{cases}$	$\begin{cases} 35^{\circ}24'48.19', \\ 41^{\circ}25'24.17', \\ 43^{\circ}9'58.40' \end{cases}$	$\begin{cases} 36^{\circ}13'44.26'', \\ 30^{\circ}13'8.23'', \\ 28^{\circ}28'34.01'' \end{cases}$	$\begin{cases} 36^{\circ}13'44.26'', \\ 30^{\circ}13'8.23'', \\ 28^{\circ}28'34.01'' \end{cases}$	25.39085	
3. $\begin{cases} a=7.4779, \\ b=8.1235, \\ c=9.2436, \end{cases}$	$\begin{cases} 18^{\circ}1'21.39', \\ 15^{\circ}1'40.71', \\ 16^{\circ}35'47.81' \end{cases}$	$\begin{cases} 35^{\circ}33'48.79', \\ 47^{\circ}39'13.97', \\ 47^{\circ}27'2.80' \end{cases}$	$\begin{cases} 26^{\circ}51'16.49', \\ 30^{\circ}22'56.90', \\ 25^{\circ}38'21.76' \end{cases}$	$\begin{cases} 32^{\circ}33'45.21'', \\ 39^{\circ}35'7.60'', \\ 39^{\circ}33'8.80'' \end{cases}$	$\begin{cases} 35^{\circ}21'19.15'', \\ 28^{\circ}19'56.74'', \\ 30^{\circ}42'38.45'' \end{cases}$	$\begin{cases} 35^{\circ}21'19.15'', \\ 28^{\circ}19'56.74'', \\ 30^{\circ}42'38.45'' \end{cases}$	25.18782	$S_1=\sigma_1=124^{\circ}0'55.55''$
4. $\begin{cases} a=3.3333, \\ b=3.1111, \\ c=4.668152, \end{cases}$	$\begin{cases} 29^{\circ}50'59.97', \\ 19^{\circ}54'42.60', \\ 29^{\circ}50'34.68' \end{cases}$	$\begin{cases} 41^{\circ}18'47.86', \\ 28^{\circ}18'50.73', \\ 47^{\circ}24'30.55' \end{cases}$	$\begin{cases} 22^{\circ}40'47.07', \\ 20^{\circ}20'49.08', \\ 25^{\circ}30'23.75' \end{cases}$	$\begin{cases} 31^{\circ}44'53.38'', \\ 27^{\circ}12'45.74'', \\ 30^{\circ}22'39.20'' \end{cases}$	$\begin{cases} 38^{\circ}56'50.18'', \\ 23^{\circ}11'2.35'', \\ 38^{\circ}55'41.93'' \end{cases}$	$\begin{cases} 38^{\circ}56'50.18'', \\ 23^{\circ}11'2.35'', \\ 38^{\circ}55'41.93'' \end{cases}$	4.370334	$S_2=\sigma_2=117^{\circ}2'15.12''$
5. $\begin{cases} a=5.2669, \\ b=7.8712, \\ c=7.8836, \end{cases}$	$\begin{cases} 28^{\circ}18'47.07', \\ 21^{\circ}36'9.84', \\ 21^{\circ}35'7.92' \end{cases}$	$\begin{cases} 45^{\circ}39'16.88'', \\ 47^{\circ}30'47.98'', \\ 47^{\circ}32'18.12'' \end{cases}$	$\begin{cases} 20^{\circ}54'4.10'', \\ 23^{\circ}13'25.39'', \\ 30^{\circ}53'7.14'' \end{cases}$	$\begin{cases} 20^{\circ}51'32.04'', \\ 30^{\circ}48'5.79'', \\ 23^{\circ}8'47.73'' \end{cases}$	$\begin{cases} 24^{\circ}27'40.25'', \\ 47^{\circ}24'21.59'', \\ 47^{\circ}25'55.78'' \end{cases}$	$\begin{cases} 43^{\circ}52'5.60'', \\ 27^{\circ}30'1.99'', \\ 27^{\circ}28'27.88'' \end{cases}$	25.49455	$S_2=\sigma_2=138^{\circ}42'22.97''$
6. $\begin{cases} a=2.6335, \\ b=2.7093, \\ c=3.8318, \end{cases}$	$\begin{cases} 44^{\circ}30'31.86'', \\ 45^{\circ}50'43.40'', \\ 45^{\circ}58'17.20'' \end{cases}$	$\begin{cases} 29^{\circ}37'2.01'', \\ 22^{\circ}24'25.69'', \\ 22^{\circ}23'19.20'' \end{cases}$	$\begin{cases} 21^{\circ}37'17.11'', \\ 30^{\circ}9'2.45'', \\ 22^{\circ}43'43.89'' \end{cases}$	$\begin{cases} 21^{\circ}39'56.95'', \\ 22^{\circ}46'44.25'', \\ 30^{\circ}13'55.51'' \end{cases}$	$\begin{cases} 26^{\circ}0'44.50'', \\ 27^{\circ}32'57.90'', \\ 27^{\circ}31'22.62'' \end{cases}$	$\begin{cases} 44^{\circ}32'25.39'', \\ 45^{\circ}55'45.88'', \\ 45^{\circ}57'21.25'' \end{cases}$	3.210399	$S_1=\sigma_1=136^{\circ}25'32.52''$
7. $\begin{cases} a=2.6155, \\ b=2.6093, \\ c=2.6031, \end{cases}$	$\begin{cases} 45^{\circ}24'13.12'', \\ 44^{\circ}9'34.68'', \\ 44^{\circ}11'18.05'' \end{cases}$	$\begin{cases} 30^{\circ}39'53.85'', \\ 22^{\circ}32'14.79'', \\ 22^{\circ}51'0.31'' \end{cases}$	$\begin{cases} 22^{\circ}56'26.50'', \\ 22^{\circ}13'52.49'', \\ 22^{\circ}13'52.49'' \end{cases}$	$\begin{cases} 22^{\circ}59'25.01'', \\ 22^{\circ}19'54.39'', \\ 29^{\circ}42'31.29'' \end{cases}$	$\begin{cases} 28^{\circ}18'19.42'', \\ 27^{\circ}21'8.69'', \\ 27^{\circ}19'26.23'' \end{cases}$	$\begin{cases} 45^{\circ}25'40.03'', \\ 44^{\circ}8'51.98'', \\ 44^{\circ}10'34.47'' \end{cases}$	2.958072	$S_1=\sigma_1=133^{\circ}45'6.48''$
8. $\begin{cases} a=4.4877, \\ b=5.4871, \\ c=6.6877, \end{cases}$	$\begin{cases} 32^{\circ}39'13.33'', \\ 32^{\circ}52'5.53'', \\ 25^{\circ}52'59.29'' \end{cases}$	$\begin{cases} 44^{\circ}56'27.89'', \\ 37^{\circ}53'42.32'', \\ 45^{\circ}3'29.68'' \end{cases}$	$\begin{cases} 26^{\circ}13'18.39'', \\ 24^{\circ}3'49.92'', \\ 33^{\circ}2'41.31'' \end{cases}$	$\begin{cases} 21^{\circ}4'24.79'', \\ 24^{\circ}18'5.36'', \\ 20^{\circ}59'42.19'' \end{cases}$	$\begin{cases} 30^{\circ}3'56.67'', \\ 37^{\circ}53'41.77'', \\ 45^{\circ}4'30.01'' \end{cases}$	$\begin{cases} 44^{\circ}55'28.11'', \\ 27^{\circ}16'14.54'', \\ 29^{\circ}54'54.77'' \end{cases}$	15.95682	$S_2=\sigma_2=127^{\circ}53'39.89''$
9. $\begin{cases} a=4.6739, \\ b=4.5887, \\ c=4.4435, \end{cases}$	$\begin{cases} 47^{\circ}11'6.93'', \\ 45^{\circ}58'14'', \\ 44^{\circ}15'14.04'' \end{cases}$	$\begin{cases} 30^{\circ}47'39.37'', \\ 22^{\circ}30'22.25'', \\ 21^{\circ}18'32.58'' \end{cases}$	$\begin{cases} 23^{\circ}14'23.60'', \\ 30^{\circ}1'43.41'', \\ 20^{\circ}40'25.06'' \end{cases}$	$\begin{cases} 22^{\circ}42'28.08'', \\ 21^{\circ}29'28.01'', \\ 29^{\circ}10'37.22'' \end{cases}$	$\begin{cases} 28^{\circ}22'50.84'', \\ 20^{\circ}35'44.75'', \\ 25^{\circ}1'14.87'' \end{cases}$	$\begin{cases} 47^{\circ}39'43.28'', \\ 45^{\circ}55'3.45'', \\ 44^{\circ}20'33.60'' \end{cases}$	15.66689	$S_1=\sigma_1=137^{\circ}55'20.33''$
10. $\begin{cases} a=8.7737, \\ b=7.8837, \\ c=7.7745, \end{cases}$	$\begin{cases} 30^{\circ}13'59.65'', \\ 31^{\circ}51'17.90'', \\ 31^{\circ}12'45.19'' \end{cases}$	$\begin{cases} 26^{\circ}52'34.55'', \\ 26^{\circ}52'26.72'', \\ 26^{\circ}17'50.37'' \end{cases}$	$\begin{cases} 31^{\circ}54'48.09'', \\ 31^{\circ}51'56.80'', \\ 29^{\circ}34'26.10'' \end{cases}$	$\begin{cases} 31^{\circ}50'51.73'', \\ 30^{\circ}13'34.25'', \\ 31^{\circ}12'28.65'' \end{cases}$	$\begin{cases} 36^{\circ}54'45.66'', \\ 34^{\circ}22'33.72'', \\ 33^{\circ}19'22.88'' \end{cases}$	$\begin{cases} 34^{\circ}22'48.48'', \\ 36^{\circ}55'0.43'', \\ 35^{\circ}51'49.59'' \end{cases}$	66.38926	$a+x=b+y, y=b-z-c$

EXAMPLES. *N. — Continued.*

DATA.	$\frac{1}{2}A$	$\frac{1}{2}B$	$\frac{1}{2}C$	$\frac{1}{2}D$	$\frac{1}{2}E$	$\frac{1}{2}F$	$\frac{1}{2}G$	$\frac{1}{2}H$	$\frac{1}{2}I$	$\frac{1}{2}J$	$\frac{1}{2}K$	$\frac{1}{2}L$	$\frac{1}{2}M$	VOLUME.	REMARKS.
11. $\begin{cases} a=5.0837, \\ b=5.5789, \\ c=0.0741, \end{cases}$ $\begin{cases} x=9.9998 \\ y=9.5046 \\ z=9.0094 \end{cases}$	59° 4' 34.92"	32° 39' 35.66"	15° 20' 34.51"	14° 17' 35.99"	10° 3' 55.74"	72° 29' 0.19"	13° 33' 528"	$a+x=b+y=z$							
12. $\begin{cases} a=9.9667, \\ b=7.5449, \\ c=7.4337, \end{cases}$ $\begin{cases} x=9.8554 \\ y=7.4336 \\ z=7.3224 \end{cases}$	43° 8' 35.07"	41° 54' 9.67"	42° 5' 20.70"	42° 5' 46.27"	63° 32' 56.01"	63° 17' 47.55"	40.56799	$a-x=b-y=z$							
13. $\begin{cases} a=4.5567, \\ b=4.1233, \\ c=3.6903, \end{cases}$ $\begin{cases} x=5.5208 \\ y=5.9630 \\ z=6.14996 \end{cases}$	44° 57' 34.70"	27° 8' 57.02"	23° 54' 19.39"	24° 53' 43.91"	28° 52' 4.34"	44° 57' 39.91"	11.54943	$a+x=b+y$ $S_1=\sigma=136.011.94$							
14. $\begin{cases} a=2.5155, \\ b=3.421484, \\ c=3.7418, \end{cases}$ $\begin{cases} x=3.8118 \\ y=2.5093 \\ z=2.5033 \end{cases}$	32° 3' 29.18"	49° 30' 11.85"	23° 35' 6.71"	20° 57' 59.31"	27° 51' 33.87"	49° 19' 1.80"	2.581718	$S_2=\sigma=140.36'46"$							
15. $\begin{cases} a=7.8530, \\ b=6.4705, \\ c=6.6899, \end{cases}$ $\begin{cases} x=9.8528 \\ y=9.1375 \\ z=8.6897 \end{cases}$	48° 28' 5.76"	33° 31' 26.50"	30° 9' 7.76"	28° 30' 7.42"	39° 26' 19.75"	50° 33' 40.25"	52.14756	$A+x=180^\circ$, vol. a max. for variations of a, x , so that $\Delta x - \Delta x = 0$							
16. $\begin{cases} a=4.3719, \\ b=5.3726, \\ c=5.6464, \end{cases}$ $\begin{cases} x=3.2608 \\ y=4.2615 \\ z=4.5353 \end{cases}$	17° 9' 20.93"	21° 41' 16.39"	35° 45' 44.77"	25° 0' 13.92"	26° 33' 13.41"	21° 55' 41.07"	9.274307	$C+Z=180^\circ$, vol. a max. for variations of c, z , such that $\Delta x - \Delta x = 0$							
17. $\begin{cases} a=2.1995, \\ b=2.3207, \\ c=2.6116, \end{cases}$ $\begin{cases} x=3.4633 \\ y=3.5845 \\ z=3.8754 \end{cases}$	44° 29' 59.11"	27° 34' 43.45"	15° 2' 4.52"	18° 48' 21.62"	17° 37' 2.56"	46° 1' 6.46"	1.949617								
18. $\begin{cases} a=4.8613, \\ b=4.9233, \\ c=5.5121, \end{cases}$ $\begin{cases} x=4.7940 \\ y=4.7320 \\ z=4.1432 \end{cases}$	27° 10' 15.48"	32° 29' 48.09"	32° 5' 12.61"	28° 1' 7.50"	35° 21' 0.99"	35° 21' 48.53"	12.85487	$A+x=B+y=C+Z$							
19. $\begin{cases} a=4.3723, \\ b=5.3723, \\ c=5.6463, \end{cases}$ $\begin{cases} x=3.2612 \\ y=4.2612 \\ z=4.5352 \end{cases}$	24° 9' 15.71"	37° 4' 51.85"	31° 43' 28.55"	28° 16' 40.79"	35° 51' 30.43"	34° 51' 19.07"	9.275219	$A-x=B-y=C-Z$							
20. $\begin{cases} a=4.8624, \\ b=4.9233, \\ c=5.5121, \end{cases}$ $\begin{cases} x=4.7929 \\ y=4.7320 \\ z=4.1432 \end{cases}$	22° 9' 50.81"	32° 29' 48.09"	32° 5' 12.66"	28° 1' 32.99"	35° 21' 24.70"	35° 21' 24.70"	12.85487	(18) differs from this by $\Delta a = -0.001$, $\Delta x = -0.001$, and since $A = x$, I is a max., and $\Delta I = 0$.							

EXAMPLES. X.—Continued.

DATA.	$\frac{1}{2}A$	$\frac{1}{3}A$	$\frac{1}{4}A$	$\frac{1}{5}A$	$\frac{1}{6}A$	$\frac{1}{7}A$	$\frac{1}{8}A$	$\frac{1}{9}A$	$\frac{1}{10}A$	$\frac{1}{11}A$	$\frac{1}{12}A$	$\frac{1}{13}A$	$\frac{1}{14}A$	$\frac{1}{15}A$	$\frac{1}{16}A$	$\frac{1}{17}A$	$\frac{1}{18}A$	$\frac{1}{19}A$	$\frac{1}{20}A$	VOLUME.	REMARKS.
21. $\left\{ \begin{array}{l} a=3.309, \\ b=7.250, \\ c=5.2790, \end{array} \right.$	$\frac{1}{2}A$ 42° 39' 49.11"	$\frac{1}{3}A$ 25° 32' 56.83"	$\frac{1}{4}A$ 25° 28' 56.83"	$\frac{1}{5}A$ 25° 28' 56.83"	$\frac{1}{6}A$ 25° 28' 56.83"	$\frac{1}{7}A$ 25° 28' 56.83"	$\frac{1}{8}A$ 25° 28' 56.83"	$\frac{1}{9}A$ 25° 28' 56.83"	$\frac{1}{10}A$ 25° 28' 56.83"	$\frac{1}{11}A$ 25° 28' 56.83"	$\frac{1}{12}A$ 25° 28' 56.83"	$\frac{1}{13}A$ 25° 28' 56.83"	$\frac{1}{14}A$ 25° 28' 56.83"	$\frac{1}{15}A$ 25° 28' 56.83"	$\frac{1}{16}A$ 25° 28' 56.83"	$\frac{1}{17}A$ 25° 28' 56.83"	$\frac{1}{18}A$ 25° 28' 56.83"	$\frac{1}{19}A$ 25° 28' 56.83"	$\frac{1}{20}A$ 25° 28' 56.83"	5.53466	$A+X=B+Y=C+Z$
22. $\left\{ \begin{array}{l} a=7.1186, \\ b=4.2250, \\ c=1.1705, \end{array} \right.$	$\frac{1}{2}A$ 16° 6' 49.54"	$\frac{1}{3}A$ 10° 4' 12.51"	$\frac{1}{4}A$ 10° 4' 12.51"	$\frac{1}{5}A$ 10° 4' 12.51"	$\frac{1}{6}A$ 10° 4' 12.51"	$\frac{1}{7}A$ 10° 4' 12.51"	$\frac{1}{8}A$ 10° 4' 12.51"	$\frac{1}{9}A$ 10° 4' 12.51"	$\frac{1}{10}A$ 10° 4' 12.51"	$\frac{1}{11}A$ 10° 4' 12.51"	$\frac{1}{12}A$ 10° 4' 12.51"	$\frac{1}{13}A$ 10° 4' 12.51"	$\frac{1}{14}A$ 10° 4' 12.51"	$\frac{1}{15}A$ 10° 4' 12.51"	$\frac{1}{16}A$ 10° 4' 12.51"	$\frac{1}{17}A$ 10° 4' 12.51"	$\frac{1}{18}A$ 10° 4' 12.51"	$\frac{1}{19}A$ 10° 4' 12.51"	$\frac{1}{20}A$ 10° 4' 12.51"	7.857833	$A+X=B+Y=C+Z$
23. $\left\{ \begin{array}{l} a=7.1186, \\ b=4.2245, \\ c=1.1705, \end{array} \right.$	$\frac{1}{2}A$ 16° 6' 57.96"	$\frac{1}{3}A$ 10° 4' 12.51"	$\frac{1}{4}A$ 10° 4' 12.51"	$\frac{1}{5}A$ 10° 4' 12.51"	$\frac{1}{6}A$ 10° 4' 12.51"	$\frac{1}{7}A$ 10° 4' 12.51"	$\frac{1}{8}A$ 10° 4' 12.51"	$\frac{1}{9}A$ 10° 4' 12.51"	$\frac{1}{10}A$ 10° 4' 12.51"	$\frac{1}{11}A$ 10° 4' 12.51"	$\frac{1}{12}A$ 10° 4' 12.51"	$\frac{1}{13}A$ 10° 4' 12.51"	$\frac{1}{14}A$ 10° 4' 12.51"	$\frac{1}{15}A$ 10° 4' 12.51"	$\frac{1}{16}A$ 10° 4' 12.51"	$\frac{1}{17}A$ 10° 4' 12.51"	$\frac{1}{18}A$ 10° 4' 12.51"	$\frac{1}{19}A$ 10° 4' 12.51"	$\frac{1}{20}A$ 10° 4' 12.51"	7.857836	$A+X=B+Y=C+Z$ $\Delta I'$ (in 22) very small
24. $\left\{ \begin{array}{l} a=7.1187, \\ b=4.2237, \\ c=1.1697, \end{array} \right.$	$\frac{1}{2}A$ 16° 7' 6.75"	$\frac{1}{3}A$ 10° 4' 12.51"	$\frac{1}{4}A$ 10° 4' 12.51"	$\frac{1}{5}A$ 10° 4' 12.51"	$\frac{1}{6}A$ 10° 4' 12.51"	$\frac{1}{7}A$ 10° 4' 12.51"	$\frac{1}{8}A$ 10° 4' 12.51"	$\frac{1}{9}A$ 10° 4' 12.51"	$\frac{1}{10}A$ 10° 4' 12.51"	$\frac{1}{11}A$ 10° 4' 12.51"	$\frac{1}{12}A$ 10° 4' 12.51"	$\frac{1}{13}A$ 10° 4' 12.51"	$\frac{1}{14}A$ 10° 4' 12.51"	$\frac{1}{15}A$ 10° 4' 12.51"	$\frac{1}{16}A$ 10° 4' 12.51"	$\frac{1}{17}A$ 10° 4' 12.51"	$\frac{1}{18}A$ 10° 4' 12.51"	$\frac{1}{19}A$ 10° 4' 12.51"	$\frac{1}{20}A$ 10° 4' 12.51"	7.857187	$A+X=B+Y=C+Z$
25. $\left\{ \begin{array}{l} a=8.8524, \\ b=7.6523, \\ c=10.6117, \end{array} \right.$	$\frac{1}{2}A$ 32° 49' 32.81"	$\frac{1}{3}A$ 25° 58' 33.28"	$\frac{1}{4}A$ 25° 58' 33.28"	$\frac{1}{5}A$ 25° 58' 33.28"	$\frac{1}{6}A$ 25° 58' 33.28"	$\frac{1}{7}A$ 25° 58' 33.28"	$\frac{1}{8}A$ 25° 58' 33.28"	$\frac{1}{9}A$ 25° 58' 33.28"	$\frac{1}{10}A$ 25° 58' 33.28"	$\frac{1}{11}A$ 25° 58' 33.28"	$\frac{1}{12}A$ 25° 58' 33.28"	$\frac{1}{13}A$ 25° 58' 33.28"	$\frac{1}{14}A$ 25° 58' 33.28"	$\frac{1}{15}A$ 25° 58' 33.28"	$\frac{1}{16}A$ 25° 58' 33.28"	$\frac{1}{17}A$ 25° 58' 33.28"	$\frac{1}{18}A$ 25° 58' 33.28"	$\frac{1}{19}A$ 25° 58' 33.28"	$\frac{1}{20}A$ 25° 58' 33.28"	69.37919	$C+Z=180^\circ$ very nearly
26. $\left\{ \begin{array}{l} a=8.8529, \\ b=7.6523, \\ c=10.6140, \end{array} \right.$	$\frac{1}{2}A$ 32° 49' 5.53"	$\frac{1}{3}A$ 25° 58' 18.85"	$\frac{1}{4}A$ 25° 58' 18.85"	$\frac{1}{5}A$ 25° 58' 18.85"	$\frac{1}{6}A$ 25° 58' 18.85"	$\frac{1}{7}A$ 25° 58' 18.85"	$\frac{1}{8}A$ 25° 58' 18.85"	$\frac{1}{9}A$ 25° 58' 18.85"	$\frac{1}{10}A$ 25° 58' 18.85"	$\frac{1}{11}A$ 25° 58' 18.85"	$\frac{1}{12}A$ 25° 58' 18.85"	$\frac{1}{13}A$ 25° 58' 18.85"	$\frac{1}{14}A$ 25° 58' 18.85"	$\frac{1}{15}A$ 25° 58' 18.85"	$\frac{1}{16}A$ 25° 58' 18.85"	$\frac{1}{17}A$ 25° 58' 18.85"	$\frac{1}{18}A$ 25° 58' 18.85"	$\frac{1}{19}A$ 25° 58' 18.85"	$\frac{1}{20}A$ 25° 58' 18.85"	69.37917	$A=X, B=Y$
27. $\left\{ \begin{array}{l} a=8.8529, \\ b=7.6514, \\ c=10.6117, \end{array} \right.$	$\frac{1}{2}A$ 32° 49' 31.54"	$\frac{1}{3}A$ 25° 58' 42.05"	$\frac{1}{4}A$ 25° 58' 42.05"	$\frac{1}{5}A$ 25° 58' 42.05"	$\frac{1}{6}A$ 25° 58' 42.05"	$\frac{1}{7}A$ 25° 58' 42.05"	$\frac{1}{8}A$ 25° 58' 42.05"	$\frac{1}{9}A$ 25° 58' 42.05"	$\frac{1}{10}A$ 25° 58' 42.05"	$\frac{1}{11}A$ 25° 58' 42.05"	$\frac{1}{12}A$ 25° 58' 42.05"	$\frac{1}{13}A$ 25° 58' 42.05"	$\frac{1}{14}A$ 25° 58' 42.05"	$\frac{1}{15}A$ 25° 58' 42.05"	$\frac{1}{16}A$ 25° 58' 42.05"	$\frac{1}{17}A$ 25° 58' 42.05"	$\frac{1}{18}A$ 25° 58' 42.05"	$\frac{1}{19}A$ 25° 58' 42.05"	$\frac{1}{20}A$ 25° 58' 42.05"	69.37920	
28. $\left\{ \begin{array}{l} a=8.8529, \\ b=7.6523, \\ c=10.6117, \end{array} \right.$	$\frac{1}{2}A$ 32° 49' 25.20"	$\frac{1}{3}A$ 25° 58' 30.21"	$\frac{1}{4}A$ 25° 58' 30.21"	$\frac{1}{5}A$ 25° 58' 30.21"	$\frac{1}{6}A$ 25° 58' 30.21"	$\frac{1}{7}A$ 25° 58' 30.21"	$\frac{1}{8}A$ 25° 58' 30.21"	$\frac{1}{9}A$ 25° 58' 30.21"	$\frac{1}{10}A$ 25° 58' 30.21"	$\frac{1}{11}A$ 25° 58' 30.21"	$\frac{1}{12}A$ 25° 58' 30.21"	$\frac{1}{13}A$ 25° 58' 30.21"	$\frac{1}{14}A$ 25° 58' 30.21"	$\frac{1}{15}A$ 25° 58' 30.21"	$\frac{1}{16}A$ 25° 58' 30.21"	$\frac{1}{17}A$ 25° 58' 30.21"	$\frac{1}{18}A$ 25° 58' 30.21"	$\frac{1}{19}A$ 25° 58' 30.21"	$\frac{1}{20}A$ 25° 58' 30.21"	69.37921	$A=X, B=Y, C=Z=180^\circ$. I' a max. subject to $a+x=17.7088, b+y=15.3046, z-c=1.6228$
29. $\left\{ \begin{array}{l} a=7.1182, \\ b=4.2394, \\ c=1.1705, \end{array} \right.$	$\frac{1}{2}A$ 16° 5' 13.33"	$\frac{1}{3}A$ 10° 4' 12.51"	$\frac{1}{4}A$ 10° 4' 12.51"	$\frac{1}{5}A$ 10° 4' 12.51"	$\frac{1}{6}A$ 10° 4' 12.51"	$\frac{1}{7}A$ 10° 4' 12.51"	$\frac{1}{8}A$ 10° 4' 12.51"	$\frac{1}{9}A$ 10° 4' 12.51"	$\frac{1}{10}A$ 10° 4' 12.51"	$\frac{1}{11}A$ 10° 4' 12.51"	$\frac{1}{12}A$ 10° 4' 12.51"	$\frac{1}{13}A$ 10° 4' 12.51"	$\frac{1}{14}A$ 10° 4' 12.51"	$\frac{1}{15}A$ 10° 4' 12.51"	$\frac{1}{16}A$ 10° 4' 12.51"	$\frac{1}{17}A$ 10° 4' 12.51"	$\frac{1}{18}A$ 10° 4' 12.51"	$\frac{1}{19}A$ 10° 4' 12.51"	$\frac{1}{20}A$ 10° 4' 12.51"	7.859762	$A+X=B+Y=C+Z$ $\Delta I' = 121^\circ 59' 43''$
30. $\left\{ \begin{array}{l} a=3.5703, \\ b=3.6707, \\ c=1.2233, \end{array} \right.$	$\frac{1}{2}A$ 35° 55' 11.38"	$\frac{1}{3}A$ 23° 45' 2.29"	$\frac{1}{4}A$ 23° 45' 2.29"	$\frac{1}{5}A$ 23° 45' 2.29"	$\frac{1}{6}A$ 23° 45' 2.29"	$\frac{1}{7}A$ 23° 45' 2.29"	$\frac{1}{8}A$ 23° 45' 2.29"	$\frac{1}{9}A$ 23° 45' 2.29"	$\frac{1}{10}A$ 23° 45' 2.29"	$\frac{1}{11}A$ 23° 45' 2.29"	$\frac{1}{12}A$ 23° 45' 2.29"	$\frac{1}{13}A$ 23° 45' 2.29"	$\frac{1}{14}A$ 23° 45' 2.29"	$\frac{1}{15}A$ 23° 45' 2.29"	$\frac{1}{16}A$ 23° 45' 2.29"	$\frac{1}{17}A$ 23° 45' 2.29"	$\frac{1}{18}A$ 23° 45' 2.29"	$\frac{1}{19}A$ 23° 45' 2.29"	$\frac{1}{20}A$ 23° 45' 2.29"	8.638628	$X' = I' - R = Z - C$ $: C - X'$

EXAMPLES. X.—Continued.

DATA.	$\frac{1}{2}a$, $\frac{1}{2}b$, $\frac{1}{2}c$	$\frac{1}{2}a$, $\frac{1}{2}b$, $\frac{1}{2}c$	$\frac{1}{2}a$, $\frac{1}{2}b$, $\frac{1}{2}c$	$\frac{1}{2}a$, $\frac{1}{2}b$, $\frac{1}{2}c$	$\frac{1}{2}a$, $\frac{1}{2}b$, $\frac{1}{2}c$	$\frac{1}{2}a$, $\frac{1}{2}b$, $\frac{1}{2}c$	VOLUME.	REMARKS.
31. $\left\{ \begin{array}{l} a=7.1182, \\ b=4.1092, \\ c=4.1697, \end{array} \right.$	$16^{\circ}21' 0.83''$ $24^{\circ} 7' 9.93''$ $23^{\circ} 38' 46.75''$	$12^{\circ}40' 35.66''$ $17^{\circ}26' 29.93''$ $17^{\circ}48' 37.61''$	$45^{\circ}54' 43.32''$ $39^{\circ}23' 18.63''$ $37^{\circ}32' 18.02''$	$48^{\circ} 4' 12.46''$ $36^{\circ} 52' 18.18''$ $36^{\circ} 36' 10.72''$	$50^{\circ}40' 1.17''$ $36^{\circ} 52' 18.18''$ $36^{\circ} 30' 8.93''$	$22^{\circ}20' 16.69''$ $38^{\circ} 3' 25.68''$ $36^{\circ} 30' 8.93''$	7.833902	$A+X=B+Y=C+Z$ $C=Z$ I^{max} for variations of c , z , such that $\Delta C+\Delta Z=0$
32. $\left\{ \begin{array}{l} a=7.1182, \\ b=4.1092, \\ c=4.1717, \end{array} \right.$	$16^{\circ}20' 44.95''$ $24^{\circ} 7' 15.23''$ $23^{\circ} 37' 55.69''$	$12^{\circ}40' 43.43''$ $17^{\circ}26' 44.55''$ $17^{\circ}49' 16.47''$	$48^{\circ}55' 39.36''$ $39^{\circ}24' 36.07''$ $37^{\circ}33' 50.95''$	$48^{\circ} 3' 28.28''$ $36^{\circ} 52' 40.10''$ $37^{\circ} 54' 40.50''$	$50^{\circ}40' 11.50''$ $34^{\circ} 55' 26.08''$ $36^{\circ} 31' 48.45''$	$22^{\circ}20' 9.30''$ $38^{\circ} 4' 54.74''$ $36^{\circ} 28' 32.35''$	7.833893	As compared with (31) $\Delta X=.0002$, $\Delta Y=-.002$, and $\Delta I=-.000009$
33. $\left\{ \begin{array}{l} a=7.1182, \\ b=4.1092, \\ c=4.1697, \end{array} \right.$	$16^{\circ}20' 55.64''$ $24^{\circ} 7' 5.87''$ $23^{\circ} 38' 42.59''$	$12^{\circ}40' 31.69''$ $17^{\circ}26' 27.26''$ $17^{\circ}48' 35.05''$	$48^{\circ}54' 30.15''$ $39^{\circ}23' 15.81''$ $37^{\circ}32' 20.86''$	$48^{\circ} 4' 19.08''$ $36^{\circ} 52' 12.46''$ $37^{\circ} 56' 12.46''$	$50^{\circ}40' 7.06''$ $34^{\circ} 56' 53.02''$ $36^{\circ} 30' 10.72''$	$22^{\circ}20' 12.47''$ $38^{\circ} 3' 26.51''$ $36^{\circ} 30' 8.81''$	7.833233	As compared with (31) $\Delta X=.0002$ — ΔY , and $\Delta I=-.000069$
34. $\left\{ \begin{array}{l} a=7.1181, \\ b=4.1093, \\ c=4.1698, \end{array} \right.$	$16^{\circ}21' 1.98''$ $24^{\circ} 7' 9.75''$ $23^{\circ} 38' 46.63''$	$12^{\circ}40' 38.53''$ $17^{\circ}26' 32.99''$ $17^{\circ}48' 40.67''$	$48^{\circ}54' 40.38''$ $39^{\circ}23' 12.06''$ $37^{\circ}32' 17.38''$	$48^{\circ} 4' 9.58''$ $36^{\circ} 52' 22.29''$ $37^{\circ} 56' 9.41''$	$50^{\circ}39' 58.69''$ $34^{\circ} 56' 52.29''$ $36^{\circ} 30' 8.71''$	$22^{\circ}20' 18.42''$ $38^{\circ} 3' 24.82''$ $36^{\circ} 30' 8.40''$	7.834277	As compared with (31) $\Delta X=.0001$ — ΔY , $\Delta I=-.0001$ — ΔZ , and $\Delta I=-.000375$
35. $\left\{ \begin{array}{l} a=2.3314, \\ b=5.3403, \\ c=5.2798, \end{array} \right.$	$4^{\circ} 5' 6.64''$ $24^{\circ} 4' 24.10''$ $24^{\circ} 8' 42.85''$	$59^{\circ}17' 30.80''$ $53^{\circ}31' 9.61''$ $53^{\circ}25' 37.71''$	$12^{\circ}20' 7.56''$ $15^{\circ}13' 46.96''$ $23^{\circ}16' 36.13''$	$12^{\circ}29' 58.19''$ $21^{\circ} 8' 17.25''$ $15^{\circ}28' 42.24''$	$14^{\circ}31' 55.25''$ $60^{\circ}27' 22.45''$ $60^{\circ}24' 33.25''$	$64^{\circ} 0' 6.71''$ $18^{\circ} 4' 39.51''$ $18^{\circ} 7' 28.71''$	4.800386	An ill-conditioned tetrahedron. $A+X=B+Y=C+Z=78^{\circ}32' 1.96''$
36. $\left\{ \begin{array}{l} a=3.5679, \\ b=3.6767, \\ c=3.7893, \end{array} \right.$	$41^{\circ}49' 5.28''$ $43^{\circ}43' 22.38''$ $45^{\circ} 52' 1.78''$	$28^{\circ}56' 15.44''$ $22^{\circ}29' 1.67''$ $24^{\circ} 22.84''$	$21^{\circ}38' 56.55''$ $29^{\circ}57' 50.95''$ $24^{\circ}34' 25.72''$	$22^{\circ}14' 11.81''$ $23^{\circ}36' 29.00''$ $31^{\circ} 5' 53.61''$	$26^{\circ} 9' 52.94''$ $28^{\circ} 0' 27.61''$ $37^{\circ} 42' 14.16''$	$41^{\circ}46' 28.39''$ $43^{\circ} 37' 3.06''$ $46^{\circ} 0' 57.63''$	8.220333	$X-A=1^{\circ}-B=Z-C$
37. $\left\{ \begin{array}{l} a=3.5683, \\ b=3.6766, \\ c=4.2220, \end{array} \right.$	$38^{\circ}54' 33.39''$ $40^{\circ}33' 15.50''$ $51^{\circ} 1' 33.44''$	$27^{\circ}31' 59.08''$ $19^{\circ}50' 1.62''$ $27^{\circ}31' 59.08''$	$10^{\circ} 8' 24.94''$ $28^{\circ}26' 49.20''$ $27^{\circ}59' 22.78''$	$21^{\circ}55' 45.42''$ $23^{\circ} 6' 3.83''$ $34^{\circ} 1' 11.72''$	$23^{\circ} 9' 55.50''$ $38^{\circ}59' 11.04''$ $52^{\circ}14' 32.82''$	$37^{\circ}43' 14.16''$ $38^{\circ}59' 11.04''$ $52^{\circ}14' 32.82''$	8.631324	$X-A=1^{\circ}-B=Z-C$; $C=X$
38. $\left\{ \begin{array}{l} a=4.9793, \\ b=7.8345, \\ c=5.3298, \end{array} \right.$	$13^{\circ}29' 34.11''$ $24^{\circ}16' 45.24''$ $23^{\circ}29' 58.05''$	$18^{\circ}21' 43.65''$ $44^{\circ} 57' 5.85''$ $46^{\circ} 0' 16.09''$	$21^{\circ}33' 46.10''$ $40^{\circ} 0' 1.15''$ $42^{\circ}32' 18.45''$	$19^{\circ}42' 58.67''$ $33^{\circ}58' 7.44''$ $31^{\circ}38' 15.20''$	$21^{\circ}53' 53.86''$ $43^{\circ}34' 17.41''$ $46^{\circ}44' 43.42''$	$18^{\circ}20' 8.16''$ $40^{\circ}18' 31.71''$ $37^{\circ} 8' 5.70''$	15.51394	$A-X=B-Y$ $B+Y=C+Z$
39. $\left\{ \begin{array}{l} a=4.9793, \\ b=5.0876, \\ c=5.6345, \end{array} \right.$	$10^{\circ}13' 31.16''$ $19^{\circ}58' 6.84''$ $18^{\circ} 6' 11.31''$	$31^{\circ}18' 50.85''$ $36^{\circ}53' 30.14''$ $39^{\circ}50' 12.48''$	$35^{\circ} 0' 18.55''$ $33^{\circ} 6' 15.51''$ $39^{\circ}32' 49.87''$	$30^{\circ}11' 40.68''$ $31^{\circ}13' 38.97''$ $35^{\circ}34' 53.64''$	$36^{\circ}24' 19.54''$ $38^{\circ}29' 20.72''$ $47^{\circ}17' 25.22''$	$32^{\circ} 8' 55.00''$ $34^{\circ}13' 56.18''$ $29^{\circ}25' 51.68''$	8.12844	$A-X=B-Y$ $B+Y=C+Z$
40. $\left\{ \begin{array}{l} a=2.4897, \\ b=9.1439, \\ c=9.175, \end{array} \right.$	$5^{\circ}27' 4.27''$ $29^{\circ}50' 3.39''$ $29^{\circ}43' 23.37''$	$6^{\circ} 5' 47.64''$ $52^{\circ}39' 53.54''$ $52^{\circ}47' 20.72''$	$7^{\circ}36' 43.09''$ $46^{\circ}20' 13.26''$ $46^{\circ}39' 33.08''$	$7^{\circ}22' 35.98''$ $37^{\circ}53' 2.65''$ $37^{\circ}33' 59.10''$	$6^{\circ}10' 29.50''$ $45^{\circ}15' 52.66''$ $46^{\circ}25' 37.45''$	$7^{\circ}36' 55.66''$ $43^{\circ}58' 26.56''$ $42^{\circ}48' 41.77''$	5.816192	$A-X=B-Y$, $B+Y=C+Z$. A most ill-conditioned tetrahedron.

EXAMPLES. X.—Continued.

DATA.	$\frac{1}{2}a$, $\frac{1}{2}b$, $\frac{1}{2}c$, $\frac{1}{2}d$	$\frac{1}{2}e$, $\frac{1}{2}f$, $\frac{1}{2}g$, $\frac{1}{2}h$	$\frac{1}{2}i$, $\frac{1}{2}j$, $\frac{1}{2}k$, $\frac{1}{2}l$	$\frac{1}{2}m$, $\frac{1}{2}n$, $\frac{1}{2}o$, $\frac{1}{2}p$	$\frac{1}{2}q$, $\frac{1}{2}r$, $\frac{1}{2}s$, $\frac{1}{2}t$	$\frac{1}{2}u$, $\frac{1}{2}v$, $\frac{1}{2}w$, $\frac{1}{2}x$	$\frac{1}{2}y$, $\frac{1}{2}z$, $\frac{1}{2}A$, $\frac{1}{2}B$, $\frac{1}{2}C$	VOLUME.	REMARKS.
41. $\left\{ \begin{array}{l} a=4.3723, \\ b=5.3723, \\ c=5.6335, \end{array} \right.$	$\left\{ \begin{array}{l} x=3.2612 \\ y=4.2612 \\ z=4.5224 \end{array} \right.$	$\left\{ \begin{array}{l} 17^{\circ} 11' 3.80'' \\ 24^{\circ} 12' 31.63'' \\ 27^{\circ} 3^{\circ} 55.40'' \end{array} \right.$	$\left\{ \begin{array}{l} 21^{\circ} 43' 52.58'' \\ 37^{\circ} 8' 43.27'' \\ 40^{\circ} 43' 33.90'' \end{array} \right.$	$\left\{ \begin{array}{l} 25^{\circ} 47' 21.33'' \\ 32^{\circ} 0' 3.40'' \\ 38^{\circ} 36' 7.28'' \end{array} \right.$	$\left\{ \begin{array}{l} 23^{\circ} 3^{\circ} 54.47'' \\ 38^{\circ} 42' 48.02'' \\ 30^{\circ} 16' 3.96'' \end{array} \right.$	$\left\{ \begin{array}{l} 26^{\circ} 37' 39.64'' \\ 38^{\circ} 42' 41.15'' \\ 47^{\circ} 42' 4.65'' \end{array} \right.$	$\left\{ \begin{array}{l} 21^{\circ} 57' 54.94'' \\ 33^{\circ} 24' 39.45'' \\ 42^{\circ} 24' 39.95'' \end{array} \right.$	9.274728	$A - X = B - Y = C - Z$
42. $\left\{ \begin{array}{l} a=3.3203, \\ b=3.4803, \\ c=3.7209, \end{array} \right.$	$\left\{ \begin{array}{l} x=5.1960 \\ y=5.3700 \\ z=5.6166 \end{array} \right.$	$\left\{ \begin{array}{l} 46^{\circ} 9' 5.70'' \\ 49^{\circ} 35' 1.48'' \\ 55^{\circ} 54' 51.48'' \end{array} \right.$	$\left\{ \begin{array}{l} 28^{\circ} 10' 58.45'' \\ 17^{\circ} 33' 26.70'' \\ 21^{\circ} 33' 15.76'' \end{array} \right.$	$\left\{ \begin{array}{l} 16^{\circ} 31' 41.82'' \\ 29^{\circ} 44' 20.25'' \\ 22^{\circ} 50' 35.97'' \end{array} \right.$	$\left\{ \begin{array}{l} 18^{\circ} 56' 52.76'' \\ 21^{\circ} 01' 12.11'' \\ 21^{\circ} 01' 18.33'' \end{array} \right.$	$\left\{ \begin{array}{l} 19^{\circ} 10' 42.44'' \\ 20^{\circ} 10' 31.47'' \\ 27^{\circ} 14' 44' 72'' \end{array} \right.$	$\left\{ \begin{array}{l} 48^{\circ} 14' 22.32'' \\ 50^{\circ} 44' 11.68'' \\ 56^{\circ} 18' 24.93'' \end{array} \right.$	6.474979	$A - X = B - Y = C - Z$
43. $\left\{ \begin{array}{l} a=4.8801, \\ b=4.9233, \\ c=5.517, \end{array} \right.$	$\left\{ \begin{array}{l} x=4.7752 \\ y=4.7436 \\ z=4.1430 \end{array} \right.$	$\left\{ \begin{array}{l} 26^{\circ} 32' 19.06'' \\ 26^{\circ} 32' 47.33'' \\ 25^{\circ} 0' 7.41'' \end{array} \right.$	$\left\{ \begin{array}{l} 32^{\circ} 21' 0.42'' \\ 32^{\circ} 46' 10.77'' \\ 34^{\circ} 58' 43.77'' \end{array} \right.$	$\left\{ \begin{array}{l} 32^{\circ} 13' 41.82'' \\ 31^{\circ} 48' 44.90'' \\ 34^{\circ} 37' 20.63'' \end{array} \right.$	$\left\{ \begin{array}{l} 28^{\circ} 8' 28.03'' \\ 28^{\circ} 19' 20.31'' \\ 23^{\circ} 50' 14.68'' \end{array} \right.$	$\left\{ \begin{array}{l} 35^{\circ} 27' 47.66'' \\ 35^{\circ} 49' 1.77'' \\ 39^{\circ} 11' 23.43'' \end{array} \right.$	$\left\{ \begin{array}{l} 35^{\circ} 14' 59.77'' \\ 34^{\circ} 53' 45.66'' \\ 31^{\circ} 31' 24.00'' \end{array} \right.$	12.85546	$A + X = B + Y = C + Z$
44. $\left\{ \begin{array}{l} a=9.41135, \\ b=8.76269, \\ c=8.07525, \end{array} \right.$	$\left\{ \begin{array}{l} x=8.83765 \\ y=7.46052 \\ z=6.18816 \end{array} \right.$	$\left\{ \begin{array}{l} 16^{\circ} 32' 11.88'' \\ 24^{\circ} 53' 46.20'' \\ 19^{\circ} 48' 21.18'' \end{array} \right.$	$\left\{ \begin{array}{l} 20^{\circ} 5' 15.62'' \\ 32^{\circ} 16' 1.72'' \\ 27^{\circ} 5' 38.78'' \end{array} \right.$	$\left\{ \begin{array}{l} 37^{\circ} 55' 37.10'' \\ 42^{\circ} 6' 23.62'' \\ 32^{\circ} 49' 1.09'' \end{array} \right.$	$\left\{ \begin{array}{l} 37^{\circ} 13' 35.02'' \\ 40^{\circ} 38' 47.03'' \\ 27^{\circ} 48' 20.76'' \end{array} \right.$	$\left\{ \begin{array}{l} 38^{\circ} 32' 5.39'' \\ 45^{\circ} 0' 0.00'' \\ 33^{\circ} 8' 42.47'' \end{array} \right.$	$\left\{ \begin{array}{l} 22^{\circ} 48' 11.85'' \\ 45^{\circ} 0' 0.00'' \\ 28^{\circ} 18' 0.18'' \end{array} \right.$	38.61827	$B = 1 = 90^{\circ}$; I' a max. for variations in b, y
45. $\left\{ \begin{array}{l} a=6.4321, \\ b=5.5433, \\ c=5.3391, \end{array} \right.$	$\left\{ \begin{array}{l} x=3.1660 \\ y=4.0548 \\ z=4.2590 \end{array} \right.$	$\left\{ \begin{array}{l} 16^{\circ} 52' 49.23'' \\ 19^{\circ} 27' 37.71'' \\ 20^{\circ} 24' 45.17'' \end{array} \right.$	$\left\{ \begin{array}{l} 22^{\circ} 20' 31.21'' \\ 29^{\circ} 9' 15.60'' \\ 27^{\circ} 54' 28.51'' \end{array} \right.$	$\left\{ \begin{array}{l} 40^{\circ} 25' 59.23'' \\ 32^{\circ} 7' 8.27'' \\ 34^{\circ} 47' 26.13'' \end{array} \right.$	$\left\{ \begin{array}{l} 42^{\circ} 37' 53.78'' \\ 38^{\circ} 19' 44.04'' \\ 35^{\circ} 32' 20.52'' \end{array} \right.$	$\left\{ \begin{array}{l} 44^{\circ} 45' 31.89'' \\ 38^{\circ} 45' 6.93'' \\ 35^{\circ} 49' 36.00'' \end{array} \right.$	$\left\{ \begin{array}{l} 26^{\circ} 53' 40.11'' \\ 32^{\circ} 54' 5.07'' \\ 35^{\circ} 49' 36.00'' \end{array} \right.$	10 51411	$C = \angle, A + X = B + Y = C + Z. I'$ a max. for variations of c, z , such that $\angle C + \angle Z = 0$
46. $\left\{ \begin{array}{l} a=7.809322, \\ b=6.001328, \\ c=8.810394, \end{array} \right.$	$\left\{ \begin{array}{l} x=8.133664 \\ y=4.412154 \\ z=6.982750 \end{array} \right.$	$\left\{ \begin{array}{l} 31^{\circ} 32' 54.01'' \\ 29^{\circ} 4' 5.27'' \\ 29^{\circ} 23' 0.72'' \end{array} \right.$	$\left\{ \begin{array}{l} 32^{\circ} 10' 56.83'' \\ 24^{\circ} 7' 6.85'' \\ 33^{\circ} 33' 26.32'' \end{array} \right.$	$\left\{ \begin{array}{l} 36^{\circ} 29' 52.44'' \\ 32^{\circ} 14' 9.63'' \\ 37^{\circ} 30' 3.51'' \end{array} \right.$	$\left\{ \begin{array}{l} 27^{\circ} 22' 28.40'' \\ 20^{\circ} 57' 2.49'' \\ 25^{\circ} 26' 23.53'' \end{array} \right.$	$\left\{ \begin{array}{l} 39^{\circ} 0' 46.96'' \\ 26^{\circ} 33' 54.19'' \\ 40^{\circ} 34' 30.21'' \end{array} \right.$	$\left\{ \begin{array}{l} 37^{\circ} 52' 29.94'' \\ 33^{\circ} 41' 24.24'' \\ 34^{\circ} 9' 57.27'' \end{array} \right.$	49.16382	
47. $\left\{ \begin{array}{l} a=7.060273, \\ b=7.375329, \\ c=7.31525, \end{array} \right.$	$\left\{ \begin{array}{l} x=7.13973 \\ y=6.4544 \\ z=8.13924 \end{array} \right.$	$\left\{ \begin{array}{l} 29^{\circ} 4' 40.45'' \\ 26^{\circ} 38' 39.03'' \\ 34^{\circ} 10' 40.52'' \end{array} \right.$	$\left\{ \begin{array}{l} 28^{\circ} 35' 42.85'' \\ 28^{\circ} 45' 4.73'' \\ 32^{\circ} 39' 12.42'' \end{array} \right.$	$\left\{ \begin{array}{l} 26^{\circ} 58' 14.75'' \\ 24^{\circ} 43' 23.96'' \\ 30^{\circ} 14' 59.77'' \end{array} \right.$	$\left\{ \begin{array}{l} 39^{\circ} 42' 8.55'' \\ 30^{\circ} 10' 19.78'' \\ 36^{\circ} 40' 53.19'' \end{array} \right.$	$\left\{ \begin{array}{l} 33^{\circ} 1' 28.12'' \\ 32^{\circ} 59' 14.62'' \\ 39^{\circ} 51' 15.35'' \end{array} \right.$	$\left\{ \begin{array}{l} 32^{\circ} 45' 55.20'' \\ 29^{\circ} 53' 54.76'' \\ 42^{\circ} 53' 11.45'' \end{array} \right.$	43.1705	$\sigma_1 = \sigma_2 = \frac{\pi}{2}$; whence $\sigma_1 - \beta_1 = \sigma_4 - \beta_4$ $\sigma_1 - \gamma_1 = \sigma_3 - \beta_3$ $\sigma_2 - \beta_2 = \sigma_5 - \beta_5$ and $\sigma_6 - \gamma_6 = \sigma_4 - \gamma_4$
48. $\left\{ \begin{array}{l} a=3.671925, \\ b=3.440879, \\ c=3.917612, \end{array} \right.$	$\left\{ \begin{array}{l} x=3.889785 \\ y=4.442154 \\ z=2.739308 \end{array} \right.$	$\left\{ \begin{array}{l} 35^{\circ} 46' 57.13'' \\ 31^{\circ} 30' 28.70'' \\ 28^{\circ} 23' 40.36'' \end{array} \right.$	$\left\{ \begin{array}{l} 30^{\circ} 5' 30.94'' \\ 31^{\circ} 30' 28.70'' \\ 28^{\circ} 23' 40.36'' \end{array} \right.$	$\left\{ \begin{array}{l} 23^{\circ} 49' 22.52'' \\ 28^{\circ} 8' 56.82'' \\ 18^{\circ} 45' 40.06'' \end{array} \right.$	$\left\{ \begin{array}{l} 23^{\circ} 49' 22.52'' \\ 28^{\circ} 8' 56.82'' \\ 18^{\circ} 45' 40.06'' \end{array} \right.$	$\left\{ \begin{array}{l} 36^{\circ} 50' 7.50'' \\ 37^{\circ} 45' 40.55'' \\ 32^{\circ} 41' 16.20'' \end{array} \right.$	$\left\{ \begin{array}{l} 35^{\circ} 15' 51.78'' \\ 45^{\circ} 0' 0.00'' \\ 24^{\circ} 54' 36'' \end{array} \right.$	5 21751	
49. $\left\{ \begin{array}{l} a=3.405534, \\ b=3.299274, \\ c=3.914063, \end{array} \right.$	$\left\{ \begin{array}{l} x=3.599451 \\ y=4.259339 \\ z=2.839558 \end{array} \right.$	$\left\{ \begin{array}{l} 29^{\circ} 34' 6.34'' \\ 35^{\circ} 23' 5.83'' \\ 32^{\circ} 22' 27.83'' \end{array} \right.$	$\left\{ \begin{array}{l} 28^{\circ} 23' 40.36'' \\ 31^{\circ} 30' 28.69'' \\ 30^{\circ} 5' 50.95'' \end{array} \right.$	$\left\{ \begin{array}{l} 33^{\circ} 27' 3.48'' \\ 24^{\circ} 56' 38.79'' \\ 20^{\circ} 39' 3.91'' \end{array} \right.$	$\left\{ \begin{array}{l} 21^{\circ} 50' 43.22'' \\ 25^{\circ} 56' 38.79'' \\ 20^{\circ} 39' 3.91'' \end{array} \right.$	$\left\{ \begin{array}{l} 34^{\circ} 8' 46.35'' \\ 37^{\circ} 45' 40.50'' \\ 35^{\circ} 15' 51.81'' \end{array} \right.$	$\left\{ \begin{array}{l} 32^{\circ} 41' 16.20'' \\ 45^{\circ} 0' 0.00'' \\ 27^{\circ} 9' 26.42'' \end{array} \right.$	4 825503	
50. $\left\{ \begin{array}{l} a=6.064344, \\ b=4.590051, \\ c=5.3391, \end{array} \right.$	$\left\{ \begin{array}{l} x=4.981241 \\ y=4.648049 \\ z=4.2590 \end{array} \right.$	$\left\{ \begin{array}{l} 28^{\circ} 54' 0.57'' \\ 23^{\circ} 47' 32.88'' \\ 22^{\circ} 1' 56.84'' \end{array} \right.$	$\left\{ \begin{array}{l} 33^{\circ} 56' 45.16'' \\ 26^{\circ} 57' 57.45'' \\ 28^{\circ} 59' 58.80'' \end{array} \right.$	$\left\{ \begin{array}{l} 41^{\circ} 0' 5.71'' \\ 29^{\circ} 53' 0.18'' \\ 32^{\circ} 30' 32.75'' \end{array} \right.$	$\left\{ \begin{array}{l} 37^{\circ} 12' 28.32'' \\ 28^{\circ} 35' 20.68'' \\ 26^{\circ} 10' 14.66'' \end{array} \right.$	$\left\{ \begin{array}{l} 47^{\circ} 41' 37.80'' \\ 30^{\circ} 8' 55.69'' \\ 32^{\circ} 50' 33.88'' \end{array} \right.$	$\left\{ \begin{array}{l} 42^{\circ} 18' 22.11'' \\ 30^{\circ} 8' 55.90'' \\ 27^{\circ} 27' 18.10'' \end{array} \right.$	13.65486	$A + X = 180^{\circ}$; $B = 1$; I' a max. if $a - x = c - z = 1.0601$ $b + y = c + z = 9.5981$

EXAMPLES. X.—Continued.

DATA.	$\frac{1}{2}a$, $\frac{1}{2}b$, $\frac{1}{2}c$	$\frac{1}{2}A$, $\frac{1}{2}B$, $\frac{1}{2}C$	$\frac{1}{2}a$, $\frac{1}{2}b$, $\frac{1}{2}c$	$\frac{1}{2}A$, $\frac{1}{2}B$, $\frac{1}{2}C$	$\frac{1}{2}a$, $\frac{1}{2}b$, $\frac{1}{2}c$	$\frac{1}{2}A$, $\frac{1}{2}B$, $\frac{1}{2}C$	VOLUME.	REMARKS.
51. $\left\{ \begin{array}{l} a=9.869433, \\ b=3.180923, \\ c=8.9763, \end{array} \right. \\ \left\{ \begin{array}{l} x=5.619933, \\ y=5.522177, \\ z=4.7263 \end{array} \right.$	$18^{\circ}36'30.32''$ $16^{\circ}49'38.16''$ $14^{\circ}13'16.43''$	$32^{\circ}59'16.79''$ $27^{\circ}45'34.61''$ $32^{\circ}8'0.48''$	$48^{\circ}1'8.96''$ $31^{\circ}54'33.53''$ $39^{\circ}22'32.12''$	$41^{\circ}2'21.36''$ $31^{\circ}40'57.56''$ $25^{\circ}5'49.68''$	$51^{\circ}16'24.88''$ $30^{\circ}52'40.93''$ $37^{\circ}9'5.81''$	$38^{\circ}43'35.12''$ $30^{\circ}52'40.93''$ $24^{\circ}36'16.05''$	31.12115	$\Delta m(50)A + \Delta = 180,$ $B = Y; A - \Delta = C - Z,$ $B + Y = C + Z$
52. $\left\{ \begin{array}{l} a=6.4141, \\ b=5.7819, \\ c=5.4637, \end{array} \right. \\ \left\{ \begin{array}{l} x=2.9120, \\ y=3.5442, \\ z=3.8624 \end{array} \right.$	$14^{\circ}55'18.00''$ $16^{\circ}45'40.62''$ $18^{\circ}13'50.42''$	$23^{\circ}1'38.22''$ $31^{\circ}24'46.04''$ $29^{\circ}10'49.07''$	$40^{\circ}21'23.54''$ $30^{\circ}36'1.81''$ $34^{\circ}30'31.22''$	$44^{\circ}3'30.31''$ $40^{\circ}34'10.78''$ $36^{\circ}22'19.97''$	$45^{\circ}31'22.33''$ $40^{\circ}39'9.91''$ $35^{\circ}32'33.55''$	$26^{\circ}34'0.21''$ $31^{\circ}18'12.63''$ $36^{\circ}24'48.99''$	8.867272	$2af + x^2 = B^2 + c^2 + y^2 + z^2;$ volume unaltered by interchange of y and z .
53. $\left\{ \begin{array}{l} a=6.4141, \\ b=5.7819, \\ c=5.4637, \end{array} \right. \\ \left\{ \begin{array}{l} x=2.9120, \\ y=3.5442, \\ z=3.8624 \end{array} \right.$	$14^{\circ}55'18.00''$ $18^{\circ}35'58.44''$ $16^{\circ}38'17.66''$	$23^{\circ}1'38.22''$ $31^{\circ}45'38.11''$ $29^{\circ}2'1.34''$	$40^{\circ}21'23.54''$ $36^{\circ}22'19.97''$ $34^{\circ}30'31.22''$	$44^{\circ}3'30.31''$ $40^{\circ}34'10.78''$ $36^{\circ}22'19.97''$	$45^{\circ}31'22.33''$ $40^{\circ}39'9.91''$ $35^{\circ}32'33.55''$	$26^{\circ}34'0.21''$ $31^{\circ}18'12.63''$ $36^{\circ}24'48.99''$	8.867270	Yol. equal to that of Δ supplementary
54. $\left\{ \begin{array}{l} a=6.4141, \\ b=3.8624, \\ c=5.4637, \end{array} \right. \\ \left\{ \begin{array}{l} x=2.9120, \\ y=5.7819, \\ z=3.5442 \end{array} \right.$	$15^{\circ}21'2.43''$ $28^{\circ}47'48.63''$ $14^{\circ}18'19.67''$	$11^{\circ}52'39.60''$ $15^{\circ}43'41.76''$ $26^{\circ}27'37.88''$	$50^{\circ}57'58.57''$ $63^{\circ}26'33.39''$ $53^{\circ}20'12.49''$	$34^{\circ}44'33.50''$ $21^{\circ}18'45.08''$ $14^{\circ}40'47.01''$	$74^{\circ}34'52.40''$ $10^{\circ}12'19.38''$ $73^{\circ}33'24.76''$	$7^{\circ}48'30.92''$ $70^{\circ}46'59.12''$ $7^{\circ}18'45.02''$	2.455306	Very small angles involved in the calculation.
55. $\left\{ \begin{array}{l} a=6.3571, \\ b=3.7073, \\ c=4.4077, \end{array} \right. \\ \left\{ \begin{array}{l} x=5.2896, \\ y=5.9394, \\ z=6.398 \end{array} \right.$	$30^{\circ}44'26.54''$ $32^{\circ}0'4.96''$ $22^{\circ}25'2.23''$	$29^{\circ}14'1.39''$ $36^{\circ}4'19.19''$ $20^{\circ}55'7.67''$	$37^{\circ}30'38.58''$ $36^{\circ}34'23.61''$ $23^{\circ}32'3.31''$	$37^{\circ}4'47.37''$ $35^{\circ}43'30.15''$ $24^{\circ}11'35.00''$	$43^{\circ}30'13.26''$ $30^{\circ}55'57.1''$ $24^{\circ}30'52.77''$	$37^{\circ}22'43.96''$ $40^{\circ}22'1.51''$ $25^{\circ}21'58.57''$	16.29656	$A + \Delta = B + Y$ $Y - B = Z - C$
56. $\left\{ \begin{array}{l} a=6.3571, \\ b=4.4077, \\ c=5.2896, \end{array} \right. \\ \left\{ \begin{array}{l} x=5.9394, \\ y=5.2896, \\ z=6.398 \end{array} \right.$	$30^{\circ}44'26.54''$ $29^{\circ}20'41.26''$ $23^{\circ}26'4.86''$	$29^{\circ}14'1.39''$ $21^{\circ}56'43.06''$ $27^{\circ}35'28.99''$	$37^{\circ}30'38.58''$ $33^{\circ}12'39.28''$ $35^{\circ}43'30.15''$	$37^{\circ}4'47.37''$ $33^{\circ}12'39.28''$ $34^{\circ}11'35.00''$	$43^{\circ}30'13.26''$ $30^{\circ}55'57.1''$ $24^{\circ}30'52.77''$	$37^{\circ}22'43.96''$ $40^{\circ}22'1.51''$ $25^{\circ}21'58.57''$	16.29656	In (55), (56) the vols. are equal; the values of Δ supplementary.
57. $\left\{ \begin{array}{l} a=7.4839, \\ b=7.2605, \\ c=6.8137, \end{array} \right. \\ \left\{ \begin{array}{l} x=3.5744, \\ y=3.3510, \\ z=3.3510 \end{array} \right.$	$11^{\circ}36'7.25''$ $15^{\circ}10'24.87''$ $13^{\circ}6'30.06''$	$29^{\circ}48'18.10''$ $36^{\circ}35'3.92''$ $32^{\circ}30'36.16''$	$40^{\circ}18'26.02''$ $33^{\circ}12'39.28''$ $34^{\circ}13'30.61''$	$42^{\circ}18'58.97''$ $41^{\circ}10'22.14''$ $26^{\circ}59'2.62''$	$44^{\circ}27'25.54''$ $43^{\circ}12'50.78''$ $35^{\circ}14'31.77''$	$31^{\circ}52'48.08''$ $34^{\circ}7'22.81''$ $27^{\circ}9'3.85''$	12.3818	$A + \Delta = B + Y; B - Y = C - Z;$ $\Delta = \sqrt{a^2 + b^2 + c^2}$
58. $\left\{ \begin{array}{l} a=7.4839, \\ b=7.2605, \\ c=6.8137, \end{array} \right. \\ \left\{ \begin{array}{l} x=3.5744, \\ y=3.3510, \\ z=3.3510 \end{array} \right.$	$14^{\circ}36'7.25''$ $13^{\circ}17'26.63''$ $14^{\circ}54'5.67''$	$29^{\circ}48'18.10''$ $35^{\circ}55'11.42''$ $32^{\circ}44'33.69''$	$40^{\circ}18'26.02''$ $26^{\circ}59'2.62''$ $34^{\circ}13'30.61''$	$42^{\circ}18'58.97''$ $41^{\circ}10'22.14''$ $33^{\circ}12'39.28''$	$44^{\circ}27'25.54''$ $43^{\circ}12'50.78''$ $35^{\circ}14'31.77''$	$31^{\circ}52'48.08''$ $27^{\circ}49'16.04''$ $33^{\circ}14'30.76''$	12.3818	Vols. of (57), (58) equal, and angle Δ supplementary.
59. $\left\{ \begin{array}{l} a=7.1184, \\ b=4.1092, \\ c=4.1717, \end{array} \right. \\ \left\{ \begin{array}{l} x=2.3403, \\ y=5.2778, \\ z=5.2778 \end{array} \right.$	$16^{\circ}20'44.95''$ $23^{\circ}7'11.19''$ $23^{\circ}37'51.92''$	$12^{\circ}40'43.41''$ $17^{\circ}46'21.6''$ $17^{\circ}49'13.89''$	$48^{\circ}55'16.21''$ $30^{\circ}24'36.08''$ $37^{\circ}34'30.95''$	$48^{\circ}55'16.21''$ $36^{\circ}5'24.10''$ $37^{\circ}54'40.51''$	$50^{\circ}40'18.81''$ $34^{\circ}55'24.00''$ $36^{\circ}31'46.79''$	$22^{\circ}20'12.36''$ $38^{\circ}4'52.72''$ $36^{\circ}28'29.93''$	7.8337865	Compared with (59) $\Delta f = -\cos \alpha, \Delta g = -\cos \beta, \Delta h = -\cos \gamma$ $\Delta f = -\cos \alpha, \Delta g = -\cos \beta, \Delta h = -\cos \gamma$
60. $\left\{ \begin{array}{l} a=7.7554, \\ b=5.32601, \\ c=5.70601, \end{array} \right. \\ \left\{ \begin{array}{l} x=6.9454, \\ y=6.51601, \\ z=5.33721 \end{array} \right.$	$31^{\circ}20'11.94''$ $27^{\circ}42'14.8''$ $20^{\circ}39'50.88''$	$35^{\circ}28'24.54''$ $32^{\circ}30'33.80''$ $23^{\circ}3'49.60''$	$36^{\circ}49'35.32''$ $31^{\circ}14'8.52''$ $23^{\circ}17'12.75''$	$39^{\circ}13'55.55''$ $34^{\circ}59'35.31''$ $23^{\circ}17'26.94''$	$46^{\circ}28'49.30''$ $36^{\circ}38'16.87''$ $24^{\circ}48'17.41''$	$43^{\circ}31'10.70''$ $33^{\circ}40'38.27''$ $23^{\circ}52'48.66''$	29.33436	$A - \Delta = B - Y; A + Y = 180;$ Δ a max. for variations of a, α, π , such that $\Delta \alpha = \Delta \pi$.

EXAMPLES. X.—Continued.

DATA.	$\frac{1}{2}a_1, \frac{1}{2}b_1, \frac{1}{2}c_1$	$\frac{1}{2}a_2, \frac{1}{2}b_2, \frac{1}{2}c_2$	$\frac{1}{2}a_3, \frac{1}{2}b_3, \frac{1}{2}c_3$	$\frac{1}{2}a_4, \frac{1}{2}b_4, \frac{1}{2}c_4$	$\frac{1}{2}A, \frac{1}{2}B, \frac{1}{2}C$	VOLUME.	REMARKS.
71. $\begin{cases} a=7.2485, \\ b=6.8205, \\ c=5.1220, \end{cases}$	$\begin{cases} 22^\circ 4'19.49", \\ 20^\circ 49'36.58", \\ 15^\circ 5'20.86", \end{cases}$	$\begin{cases} 33^\circ 10'59.37", \\ 22^\circ 25'39.39", \\ 22^\circ 25'39.39", \end{cases}$	$\begin{cases} 40^\circ 40'37.22", \\ 43^\circ 37'40.61", \\ 24^\circ 17'59.87", \end{cases}$	$\begin{cases} 48^\circ 49'59.22", \\ 43^\circ 37'40.61", \\ 24^\circ 17'59.87", \end{cases}$	$\begin{cases} 38^\circ 48'19.55", \\ 34^\circ 22'45.17", \\ 22^\circ 24'48.43", \end{cases}$	13 77206	
72. $\begin{cases} a=7.2485, \\ b=6.8205, \\ c=5.1220, \end{cases}$	$\begin{cases} 22^\circ 4'19.49", \\ 14^\circ 18'6.32", \\ 19^\circ 59'26.14", \end{cases}$	$\begin{cases} 33^\circ 10'59.37", \\ 32^\circ 36'52.49", \\ 20^\circ 51'11.06", \end{cases}$	$\begin{cases} 37^\circ 23'11.37", \\ 22^\circ 37'3.57", \\ 24^\circ 17'59.87", \end{cases}$	$\begin{cases} 54^\circ 50'42.62", \\ 43^\circ 37'40.61", \\ 34^\circ 11'57.00", \end{cases}$	$\begin{cases} 40^\circ 48'28.45", \\ 21^\circ 25'25.05", \\ 32^\circ 56'27.75", \end{cases}$	12 84365	
73. $\begin{cases} a=7.2485, \\ b=6.8205, \\ c=5.1220, \end{cases}$	$\begin{cases} 22^\circ 6'5.66", \\ 22^\circ 35'39.31", \\ 15^\circ 5'20.86", \end{cases}$	$\begin{cases} 31^\circ 26'51.14", \\ 34^\circ 14'11.02", \\ 20^\circ 49'36.58", \end{cases}$	$\begin{cases} 40^\circ 40'37.22", \\ 35^\circ 41'30.25", \\ 22^\circ 31'9.65", \end{cases}$	$\begin{cases} 49^\circ 44'44.03", \\ 45^\circ 22'11.09", \\ 21^\circ 48'38.61", \end{cases}$	$\begin{cases} 36^\circ 27'59.00", \\ 37^\circ 37'46.06", \\ 21^\circ 47'8.56", \end{cases}$	13-47945	
74. $\begin{cases} a=7.2485, \\ b=6.8205, \\ c=5.1220, \end{cases}$	$\begin{cases} 22^\circ 6'5.66", \\ 13^\circ 31'58.03", \\ 21^\circ 16'55.00", \end{cases}$	$\begin{cases} 31^\circ 26'51.14", \\ 32^\circ 7'34.19", \\ 18^\circ 21'32.60", \end{cases}$	$\begin{cases} 36^\circ 35'30.81", \\ 31^\circ 48'38.61", \\ 22^\circ 31'9.65", \end{cases}$	$\begin{cases} 58^\circ 3'29.31", \\ 45^\circ 22'44.69", \\ 30^\circ 14'30.35", \end{cases}$	$\begin{cases} 39^\circ 14'51.79", \\ 10^\circ 57'32.53", \\ 35^\circ 51'56.23", \end{cases}$	12 03492	(71) to (84) are also tetrahedra with the same six lengths for their edges. The six tetrahedra in which the two longest edges are opposite each other are all impossible.
75. $\begin{cases} a=7.2485, \\ b=6.8205, \\ c=5.1220, \end{cases}$	$\begin{cases} 20^\circ 53'58.77", \\ 20^\circ 51'11.06", \\ 19^\circ 59'26.14", \end{cases}$	$\begin{cases} 28^\circ 30'37.55", \\ 32^\circ 36'52.49", \\ 14^\circ 18'6.32", \end{cases}$	$\begin{cases} 37^\circ 23'11.37", \\ 32^\circ 18'35.14", \\ 15^\circ 32'49.76", \end{cases}$	$\begin{cases} 54^\circ 50'42.62", \\ 53^\circ 33'1.47", \\ 33^\circ 18'35.14", \end{cases}$	$\begin{cases} 51^\circ 28'55.49", \\ 49^\circ 13'2.84", \\ 15^\circ 43'5.96", \end{cases}$	11.86436	
76. $\begin{cases} a=7.2485, \\ b=6.8205, \\ c=5.1220, \end{cases}$	$\begin{cases} 20^\circ 53'58.77", \\ 18^\circ 21'32.66", \\ 21^\circ 16'55.00", \end{cases}$	$\begin{cases} 28^\circ 30'37.55", \\ 32^\circ 36'52.49", \\ 15^\circ 34'58.03", \end{cases}$	$\begin{cases} 37^\circ 23'11.37", \\ 29^\circ 10'47.31", \\ 15^\circ 32'49.76", \end{cases}$	$\begin{cases} 58^\circ 3'29.31", \\ 53^\circ 33'1.47", \\ 33^\circ 18'35.14", \end{cases}$	$\begin{cases} 55^\circ 39'49.09", \\ 48^\circ 44'29.8", \\ 15^\circ 5'6.60", \end{cases}$	11.34094	
77. $\begin{cases} a=7.2485, \\ b=6.8205, \\ c=5.1220, \end{cases}$	$\begin{cases} 22^\circ 28'7.17", \\ 14^\circ 18'6.32", \\ 19^\circ 40'57.73", \end{cases}$	$\begin{cases} 34^\circ 11'57.06", \\ 32^\circ 43'11.30", \\ 20^\circ 51'11.06", \end{cases}$	$\begin{cases} 37^\circ 35'20.97", \\ 22^\circ 37'3.57", \\ 24^\circ 16'36.86", \end{cases}$	$\begin{cases} 54^\circ 50'42.62", \\ 43^\circ 15'15.97", \\ 33^\circ 10'59.37", \end{cases}$	$\begin{cases} 56^\circ 38'17.11", \\ 38^\circ 26'2.57", \\ 22^\circ 42'35.79", \end{cases}$	12 7562	
78. $\begin{cases} a=7.2485, \\ b=6.8205, \\ c=5.1220, \end{cases}$	$\begin{cases} 22^\circ 31'9.66", \\ 22^\circ 23'55.02", \\ 15^\circ 5'20.86", \end{cases}$	$\begin{cases} 32^\circ 1'22.3", \\ 34^\circ 14'11.02", \\ 20^\circ 25'47.66", \end{cases}$	$\begin{cases} 40^\circ 40'37.22", \\ 36^\circ 17'43.87", \\ 22^\circ 6'5.66", \end{cases}$	$\begin{cases} 47^\circ 10'17.32", \\ 45^\circ 22'44.69", \\ 21^\circ 40'53.76", \end{cases}$	$\begin{cases} 37^\circ 22'19.32", \\ 36^\circ 56'27.33", \\ 21^\circ 38'0.51", \end{cases}$	13-39033	
79. $\begin{cases} a=7.2485, \\ b=6.8205, \\ c=5.1220, \end{cases}$	$\begin{cases} 22^\circ 31'9.66", \\ 23^\circ 21'3.20", \\ 21^\circ 16'55.00", \end{cases}$	$\begin{cases} 32^\circ 1'22.3", \\ 37^\circ 34'19", \\ 17^\circ 44'30.19", \end{cases}$	$\begin{cases} 36^\circ 35'30.81", \\ 21^\circ 40'53.76", \\ 22^\circ 6'5.66", \end{cases}$	$\begin{cases} 58^\circ 3'29.31", \\ 45^\circ 22'44.69", \\ 36^\circ 17'43.87", \end{cases}$	$\begin{cases} 50^\circ 39'8.08", \\ 40^\circ 12'2.21", \\ 16^\circ 47'31.02", \end{cases}$	11.7694	
80. $\begin{cases} a=7.2485, \\ b=6.8205, \\ c=5.1220, \end{cases}$	$\begin{cases} 21^\circ 28'4.14", \\ 20^\circ 51'11.06", \\ 19^\circ 40'57.73", \end{cases}$	$\begin{cases} 29^\circ 10'47.31", \\ 32^\circ 43'11.30", \\ 14^\circ 18'6.32", \end{cases}$	$\begin{cases} 37^\circ 35'20.97", \\ 32^\circ 18'35.14", \\ 15^\circ 40'18.85", \end{cases}$	$\begin{cases} 54^\circ 50'42.62", \\ 52^\circ 31'37.01", \\ 28^\circ 30'37.55", \end{cases}$	$\begin{cases} 51^\circ 47'4.39", \\ 48^\circ 10'54.20", \\ 32^\circ 38'55.92", \end{cases}$	11.9202	

EXAMPLES. X.—Continued.

DATA.	$\frac{1}{2}a_1, \frac{1}{2}b_1, \frac{1}{2}c_1$	$\frac{1}{2}a_2, \frac{1}{2}b_2, \frac{1}{2}c_2$	$\frac{1}{2}a_3, \frac{1}{2}b_3, \frac{1}{2}c_3$	$\frac{1}{2}a_4, \frac{1}{2}b_4, \frac{1}{2}c_4$	$\frac{1}{2}A, \frac{1}{2}B, \frac{1}{2}C$	$\frac{1}{2}V, \frac{1}{2}Z$	VOLUME.	REMARKS.
91. $\begin{cases} a=7.2485, \\ b=5.1220, \\ c=5.3556, \end{cases}$	$\begin{cases} 40^{\circ}27' 9.94'' \\ 20^{\circ}29' 30.39'' \\ 20^{\circ}49' 36.58'' \end{cases}$	$\begin{cases} 45^{\circ}22' 44.69'' \\ 22^{\circ}55' 39.39'' \\ 23^{\circ}59' 18.49'' \end{cases}$	$\begin{cases} 40^{\circ}44' 44.03'' \\ 22^{\circ}6' 5.00'' \\ 23^{\circ}36' 58.45'' \end{cases}$	$\begin{cases} 79^{\circ}30' 21.87'' \\ 7^{\circ}27' 45.82'' \\ 7^{\circ}52' 12.49'' \end{cases}$	$\begin{cases} 79^{\circ}30' 21.87'' \\ 7^{\circ}27' 45.82'' \\ 7^{\circ}52' 12.49'' \end{cases}$	$\begin{cases} 5.305218 \\ 5.305218 \\ 5.305218 \end{cases}$	See previous page.	
92. $\begin{cases} a=7.3871, \\ b=5.9835, \\ c=6.1635, \end{cases}$	$\begin{cases} 35^{\circ}10' 9.55'' \\ 25^{\circ}37' 31.80'' \\ 24^{\circ}52' 47.74'' \end{cases}$	$\begin{cases} 36^{\circ}33' 34.19'' \\ 26^{\circ}5' 8.05'' \\ 20^{\circ}51' 23.75'' \end{cases}$	$\begin{cases} 39^{\circ}2' 4.21'' \\ 27^{\circ}22' 44.36'' \\ 28^{\circ}5' 8.00'' \end{cases}$	$\begin{cases} 37^{\circ}31' 4.45'' \\ 26^{\circ}44' 11.79'' \\ 26^{\circ}3' 41.45'' \end{cases}$	$\begin{cases} 49^{\circ}51' 29.66'' \\ 27^{\circ}40' 55.35'' \\ 26^{\circ}49' 13.82'' \end{cases}$	$\begin{cases} 26.76496 \\ 26.76496 \\ 26.76496 \end{cases}$		
93. $\begin{cases} a=7.1471, \\ b=5.9669, \\ c=6.1635, \end{cases}$	$\begin{cases} 33^{\circ}46' 57.23'' \\ 20^{\circ}17' 5.52'' \\ 25^{\circ}35' 23.03'' \end{cases}$	$\begin{cases} 35^{\circ}7' 27.48'' \\ 20^{\circ}53' 1.12'' \\ 27^{\circ}36' 10.53'' \end{cases}$	$\begin{cases} 37^{\circ}31' 35.55'' \\ 28^{\circ}7' 14.44'' \\ 28^{\circ}45' 9.39'' \end{cases}$	$\begin{cases} 46^{\circ}37' 29.15'' \\ 30^{\circ}23' 35.41'' \\ 30^{\circ}22' 39.48'' \end{cases}$	$\begin{cases} 44^{\circ}58' 17.27'' \\ 29^{\circ}36' 31.07'' \\ 28^{\circ}43' 27.60'' \end{cases}$	$\begin{cases} 27.04064 \\ 27.04064 \\ 27.04064 \end{cases}$	(92) to (96) exemplify the process of finding the tetrahedron of maximum volume when $a-b, b+y, c$, and c are given.	
94. $\begin{cases} a=7.0871, \\ b=5.9769, \\ c=6.1635, \end{cases}$	$\begin{cases} 33^{\circ}24' 43.34'' \\ 26^{\circ}24' 9.16'' \\ 25^{\circ}45' 7.03'' \end{cases}$	$\begin{cases} 34^{\circ}48' 5.32'' \\ 27^{\circ}7' 43.21'' \\ 27^{\circ}48' 9.75'' \end{cases}$	$\begin{cases} 37^{\circ}7' 10.76'' \\ 28^{\circ}15' 20.69'' \\ 28^{\circ}57' 40.82'' \end{cases}$	$\begin{cases} 45^{\circ}52' 4.32'' \\ 30^{\circ}18' 33.44'' \\ 30^{\circ}49' 22.42'' \end{cases}$	$\begin{cases} 44^{\circ}12' 24.45'' \\ 30^{\circ}0' 31.53'' \\ 29^{\circ}9' 42.55'' \end{cases}$	$\begin{cases} 27.11766 \\ 27.11766 \\ 27.11766 \end{cases}$		
95. $\begin{cases} a=7.0841, \\ b=5.9807, \\ c=6.1635, \end{cases}$	$\begin{cases} 33^{\circ}23' 1.17'' \\ 26^{\circ}23' 32.04'' \\ 25^{\circ}45' 19.88'' \end{cases}$	$\begin{cases} 34^{\circ}47' 49.00'' \\ 27^{\circ}9' 27.60'' \\ 27^{\circ}49' 4.82'' \end{cases}$	$\begin{cases} 37^{\circ}5' 12.32'' \\ 28^{\circ}14' 41.58'' \\ 28^{\circ}57' 44.80'' \end{cases}$	$\begin{cases} 45^{\circ}49' 45.62'' \\ 30^{\circ}0' 48.71'' \\ 30^{\circ}50' 36.71'' \end{cases}$	$\begin{cases} 44^{\circ}10' 15.62'' \\ 30^{\circ}0' 52.12'' \\ 29^{\circ}11' 4.12'' \end{cases}$	$\begin{cases} 27.117785 \\ 27.117785 \\ 27.117785 \end{cases}$		
96. $\begin{cases} a=7.084115, \\ b=5.980813, \\ c=6.1635, \end{cases}$	$\begin{cases} 33^{\circ}23' 0.26'' \\ 26^{\circ}23' 29.92'' \\ 25^{\circ}45' 19.18'' \end{cases}$	$\begin{cases} 34^{\circ}47' 50.76'' \\ 27^{\circ}9' 29.52'' \\ 27^{\circ}49' 5.29'' \end{cases}$	$\begin{cases} 37^{\circ}5' 11.30'' \\ 28^{\circ}14' 39.56'' \\ 28^{\circ}57' 43.80'' \end{cases}$	$\begin{cases} 45^{\circ}49' 46.14'' \\ 30^{\circ}0' 50.02'' \\ 30^{\circ}50' 36.16'' \end{cases}$	$\begin{cases} 44^{\circ}10' 13.86'' \\ 30^{\circ}0' 50.02'' \\ 29^{\circ}11' 3.88'' \end{cases}$	$\begin{cases} 27.117785 \\ 27.117785 \\ 27.117785 \end{cases}$		
97. $\begin{cases} a=4.886136, \\ b=3.967768, \\ c=4.3557, \end{cases}$	$\begin{cases} 26^{\circ}58' 18.90'' \\ 22^{\circ}33' 53.45'' \\ 20^{\circ}28' 30.21'' \end{cases}$	$\begin{cases} 33^{\circ}40' 27.35'' \\ 26^{\circ}59' 59.41'' \\ 29^{\circ}33' 10.32'' \end{cases}$	$\begin{cases} 42^{\circ}31' 24.38'' \\ 30^{\circ}35' 32.63'' \\ 33^{\circ}56' 15.88'' \end{cases}$	$\begin{cases} 48^{\circ}26' 47.83'' \\ 30^{\circ}14' 47.50'' \\ 33^{\circ}41' 35.39'' \end{cases}$	$\begin{cases} 41^{\circ}33' 12.17'' \\ 23^{\circ}29' 14.39'' \\ 26^{\circ}47' 59.73'' \end{cases}$	$\begin{cases} 6.445679 \\ 6.445679 \\ 6.445679 \end{cases}$	$A-\lambda=C-Z, B+y=C+Z, A+\lambda=180, B-y=0, a$ similar case of maximum volume.	
98. $\begin{cases} a=5.3147, \\ b=4.0357, \\ c=4.3557, \end{cases}$	$\begin{cases} 30^{\circ}9' 27.67'' \\ 20^{\circ}35' 57.40'' \\ 18^{\circ}33' 42.30'' \end{cases}$	$\begin{cases} 38^{\circ}35' 14.65'' \\ 24^{\circ}32' 54.36'' \\ 27^{\circ}4' 4.68'' \end{cases}$	$\begin{cases} 49^{\circ}53' 23.63'' \\ 31^{\circ}41' 24.26'' \\ 31^{\circ}41' 24.26'' \end{cases}$	$\begin{cases} 58^{\circ}15' 33.72'' \\ 23^{\circ}29' 14.39'' \\ 26^{\circ}38' 44.13'' \end{cases}$	$\begin{cases} 52^{\circ}39' 1.48'' \\ 23^{\circ}29' 14.39'' \\ 21^{\circ}2' 11.89'' \end{cases}$	$\begin{cases} 5.912593 \\ 5.912593 \\ 5.912593 \end{cases}$	$A-\lambda=C-Z, B+y=C+Z$	
99. $\begin{cases} a=11.99015, \\ b=12.0002, \\ c=5.01087, \end{cases}$	$\begin{cases} 23^{\circ}7' 58.61'' \\ 20^{\circ}17' 26.61'' \\ 26^{\circ}30' 47.00'' \end{cases}$	$\begin{cases} 32^{\circ}15' 47.00'' \\ 31^{\circ}46' 0.51'' \\ 33^{\circ}16' 10.71'' \end{cases}$	$\begin{cases} 34^{\circ}31' 12.49'' \\ 29^{\circ}49' 11.31'' \\ 40^{\circ}41' 11.19'' \end{cases}$	$\begin{cases} 31^{\circ}35' 11.82'' \\ 37^{\circ}10' 35'' \\ 49^{\circ}40' 40.22'' \end{cases}$	$\begin{cases} 31^{\circ}35' 16.83'' \\ 26^{\circ}43' 47.56'' \\ 40^{\circ}19' 19.78'' \end{cases}$	$\begin{cases} 166.9554 \\ 166.9554 \\ 166.9554 \end{cases}$	Tetrahedron of max. vol. when $a+x, b, y$, and $c-z$ are given.	
100. $\begin{cases} a=5.96697, \\ b=6.028293, \\ c=7.5063, \end{cases}$	$\begin{cases} 23^{\circ}6' 38.24'' \\ 20^{\circ}17' 45.64'' \\ 21^{\circ}17' 22.23'' \end{cases}$	$\begin{cases} 36^{\circ}14' 24.17'' \\ 34^{\circ}43' 53.23'' \\ 43^{\circ}26' 9.60'' \end{cases}$	$\begin{cases} 33^{\circ}58' 44.51'' \\ 29^{\circ}8' 23.94'' \\ 40^{\circ}37' 17.00'' \end{cases}$	$\begin{cases} 26^{\circ}16' 4.76'' \\ 26^{\circ}16' 4.76'' \\ 24^{\circ}37' 11.89'' \end{cases}$	$\begin{cases} 34^{\circ}43' 13.30'' \\ 33^{\circ}43' 13.30'' \\ 46^{\circ}38' 17.92'' \end{cases}$	$\begin{cases} 18.889605 \\ 18.889605 \\ 18.889605 \end{cases}$	$a=b, A=B$	

EXAMPLES. X.—Continued.

DATA.	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}	λ_{11}	λ_{12}	VOLUME.	REMARKS.
111. $\left\{ \begin{array}{l} a=5.6794, \\ b=5.6318, \\ c=4.2893, \end{array} \right.$	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}	λ_{11}	λ_{12}	10.14892	
112. $\left\{ \begin{array}{l} a=5.6794, \\ b=5.6318, \\ c=4.2893, \end{array} \right.$	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}	λ_{11}	λ_{12}	10.17192	
113. $\left\{ \begin{array}{l} a=5.6794, \\ b=5.6318, \\ c=4.2893, \end{array} \right.$	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}	λ_{11}	λ_{12}	10.06527	
114. $\left\{ \begin{array}{l} a=5.6794, \\ b=5.6318, \\ c=4.2893, \end{array} \right.$	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}	λ_{11}	λ_{12}	10.10116	
115. $\left\{ \begin{array}{l} a=5.6794, \\ b=5.6318, \\ c=4.2893, \end{array} \right.$	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}	λ_{11}	λ_{12}	9.94747	
116. $\left\{ \begin{array}{l} a=5.6794, \\ b=5.6318, \\ c=4.2893, \end{array} \right.$	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}	λ_{11}	λ_{12}	9.934566	
117. $\left\{ \begin{array}{l} a=5.6794, \\ b=5.6318, \\ c=4.2893, \end{array} \right.$	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}	λ_{11}	λ_{12}	10.13696	
118. $\left\{ \begin{array}{l} a=5.6794, \\ b=5.6318, \\ c=4.2893, \end{array} \right.$	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}	λ_{11}	λ_{12}	10.13696	
119. $\left\{ \begin{array}{l} a=5.6794, \\ b=5.6318, \\ c=4.2893, \end{array} \right.$	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}	λ_{11}	λ_{12}	9.956056	In (107), (119), (120) $I_1 = I_2 = I_3$ $A_1 = B_1$ $A_2 + A_3 = 180^\circ$ $R_1 + R_2 = 180^\circ$
120. $\left\{ \begin{array}{l} a=5.6794, \\ b=5.6318, \\ c=4.2893, \end{array} \right.$	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}	λ_{11}	λ_{12}	9.956056	

EXAMPLES. X.—Continued.

DATA.	$\frac{1}{2}A$	$\frac{1}{2}B$	$\frac{1}{2}C$	$\frac{1}{2}D$	$\frac{1}{2}E$	$\frac{1}{2}F$	$\frac{1}{2}G$	$\frac{1}{2}H$	$\frac{1}{2}I$	$\frac{1}{2}J$	VOLUME.	REMARKS.
$\begin{cases} a=5.6794, \\ b=5.6318, \\ c=3.9794. \end{cases}$ 121.	$24^{\circ} 1' 31.91''$	$30' 16'' 2.70''$	$34^{\circ} 9' 9.91''$	$45^{\circ} 14' 28.04''$	$46^{\circ} 1' 19.38''$	$35^{\circ} 17' 35.48''$	$43^{\circ} 13' 42.22''$	$43^{\circ} 31' 57.71''$	$24^{\circ} 1' 14.15''$	$31^{\circ} 25' 45.34''$	9.9345675	In (116), (121), (122) $I' = I = 180^{\circ}$ $B_1 + B_2 = 180^{\circ}$
$\begin{cases} a=5.6794, \\ b=5.6318, \\ c=3.9794. \end{cases}$ 122.	$24^{\circ} 1' 31.91''$	$30' 16'' 2.70''$	$34^{\circ} 9' 9.91''$	$45^{\circ} 14' 28.04''$	$46^{\circ} 1' 19.38''$	$35^{\circ} 17' 35.48''$	$43^{\circ} 13' 42.22''$	$43^{\circ} 31' 57.71''$	$24^{\circ} 1' 14.15''$	$31^{\circ} 25' 45.34''$	9.9345675	In (117), (118) the volumes are equal and the values of A supplementary.
$\begin{cases} a=5.6794, \\ b=5.6318, \\ c=3.9794. \end{cases}$ 123.	$24^{\circ} 1' 31.91''$	$30' 16'' 2.70''$	$34^{\circ} 9' 9.91''$	$45^{\circ} 14' 28.04''$	$46^{\circ} 1' 19.38''$	$35^{\circ} 17' 35.48''$	$43^{\circ} 13' 42.22''$	$43^{\circ} 31' 57.71''$	$24^{\circ} 1' 14.15''$	$31^{\circ} 25' 45.34''$	9.9345675	
$\begin{cases} a=5.6794, \\ b=5.6318, \\ c=3.9794. \end{cases}$ 124.	$24^{\circ} 1' 31.91''$	$30' 16'' 2.70''$	$34^{\circ} 9' 9.91''$	$45^{\circ} 14' 28.04''$	$46^{\circ} 1' 19.38''$	$35^{\circ} 17' 35.48''$	$43^{\circ} 13' 42.22''$	$43^{\circ} 31' 57.71''$	$24^{\circ} 1' 14.15''$	$31^{\circ} 25' 45.34''$	9.9345675	
$\begin{cases} a=5.6794, \\ b=5.6318, \\ c=3.9794. \end{cases}$ 125.	$24^{\circ} 1' 31.91''$	$30' 16'' 2.70''$	$34^{\circ} 9' 9.91''$	$45^{\circ} 14' 28.04''$	$46^{\circ} 1' 19.38''$	$35^{\circ} 17' 35.48''$	$43^{\circ} 13' 42.22''$	$43^{\circ} 31' 57.71''$	$24^{\circ} 1' 14.15''$	$31^{\circ} 25' 45.34''$	9.9345675	
$\begin{cases} a=5.6794, \\ b=5.6318, \\ c=3.9794. \end{cases}$ 126.	$24^{\circ} 1' 31.91''$	$30' 16'' 2.70''$	$34^{\circ} 9' 9.91''$	$45^{\circ} 14' 28.04''$	$46^{\circ} 1' 19.38''$	$35^{\circ} 17' 35.48''$	$43^{\circ} 13' 42.22''$	$43^{\circ} 31' 57.71''$	$24^{\circ} 1' 14.15''$	$31^{\circ} 25' 45.34''$	9.9345675	
$\begin{cases} a=5.6794, \\ b=5.6318, \\ c=3.9794. \end{cases}$ 127.	$24^{\circ} 1' 31.91''$	$30' 16'' 2.70''$	$34^{\circ} 9' 9.91''$	$45^{\circ} 14' 28.04''$	$46^{\circ} 1' 19.38''$	$35^{\circ} 17' 35.48''$	$43^{\circ} 13' 42.22''$	$43^{\circ} 31' 57.71''$	$24^{\circ} 1' 14.15''$	$31^{\circ} 25' 45.34''$	9.9345675	
$\begin{cases} a=5.6794, \\ b=5.6318, \\ c=3.9794. \end{cases}$ 128.	$24^{\circ} 1' 31.91''$	$30' 16'' 2.70''$	$34^{\circ} 9' 9.91''$	$45^{\circ} 14' 28.04''$	$46^{\circ} 1' 19.38''$	$35^{\circ} 17' 35.48''$	$43^{\circ} 13' 42.22''$	$43^{\circ} 31' 57.71''$	$24^{\circ} 1' 14.15''$	$31^{\circ} 25' 45.34''$	9.9345675	
$\begin{cases} a=5.6794, \\ b=5.6318, \\ c=3.9794. \end{cases}$ 129.	$24^{\circ} 1' 31.91''$	$30' 16'' 2.70''$	$34^{\circ} 9' 9.91''$	$45^{\circ} 14' 28.04''$	$46^{\circ} 1' 19.38''$	$35^{\circ} 17' 35.48''$	$43^{\circ} 13' 42.22''$	$43^{\circ} 31' 57.71''$	$24^{\circ} 1' 14.15''$	$31^{\circ} 25' 45.34''$	9.9345675	
$\begin{cases} a=5.6794, \\ b=5.6318, \\ c=3.9794. \end{cases}$ 130.	$24^{\circ} 1' 31.91''$	$30' 16'' 2.70''$	$34^{\circ} 9' 9.91''$	$45^{\circ} 14' 28.04''$	$46^{\circ} 1' 19.38''$	$35^{\circ} 17' 35.48''$	$43^{\circ} 13' 42.22''$	$43^{\circ} 31' 57.71''$	$24^{\circ} 1' 14.15''$	$31^{\circ} 25' 45.34''$	9.9345675	

EXAMPLES. X.—Continued.

DATA.		$\frac{1}{2}A$	$\frac{1}{2}B$	$\frac{1}{2}C$	$\frac{1}{2}a$	$\frac{1}{2}b$	$\frac{1}{2}c$	$\frac{1}{2}A$	$\frac{1}{2}B$	$\frac{1}{2}C$	$\frac{1}{2}a$	$\frac{1}{2}b$	$\frac{1}{2}c$	VOLUME.	REMARKS.
181.	$\begin{cases} a=9.4826, & x=7.5998 \\ b=8.5612, & y=8.0736 \\ c=6.1874, & z=4.8370 \end{cases}$	29° 44' 2.48"	33° 22' 4.13"	42° 39' 19.01"	41° 6' 12.48"	48° 3' 55.23"	41° 57' 5.90"	35.69917	In these three tetra- hedra $a-x=1.88:8$ $b-y=16.634:8$ $c-z=4.8370$ and (181) is the one of maximum volume under these condi- tions, having $A+X$ $=180^\circ$, $B-Z=6^\circ$.						
	$\begin{cases} a=0.482093, & x=7.599293 \\ b=8.561929, & y=8.072871 \\ c=6.1874, & z=4.8370 \end{cases}$	29° 44' 4.84"	33° 22' 43.98"	42° 38' 52.13"	41° 6' 13.45"	48° 3' 54.77"	41° 56' 38.85"	35.69918							
182.	$\begin{cases} a=9.482061, & x=7.599261 \\ b=8.561973, & y=8.072827 \\ c=6.1874, & z=4.8370 \end{cases}$	29° 43' 43.78"	33° 22' 44.06"	42° 38' 50.46"	41° 6' 13.52"	48° 3' 52.86"	41° 56' 37.14"	35.69918							
	$\begin{cases} a=5.239019, & x=5.239909 \\ b=3.806876, & y=3.906688 \\ c=3.506886, & z=3.442099 \end{cases}$	32° 45' 54.94"	32° 50' 15.80"	33° 8' 25.61"	32° 32' 45.13"	36° 34' 38.90"	36° 34' 22.10"	11.25615	$\sigma_1=\sigma_2=\sigma_3=90^\circ$, $C+Z=180^\circ$, $\sigma_1-\beta_1=\sigma_2-\beta_2=\sigma_3-\beta_3$, $\sigma_1-\gamma_1=\sigma_2-\gamma_2=\sigma_3-\gamma_3$						
183.	$\begin{cases} a=4.600696, & x=4.705 \\ b=3.763106, & y=4.681051 \\ c=5.391406, & z=4.616923 \end{cases}$	29° 20' 28.02"	30° 23' 49.33"	32° 50' 23.27"	26° 53' 54.08"	34° 17' 7.52"	33° 41' 24.26"	10.78158	$\sigma_1=\sigma_2=\sigma_3=90^\circ$, $C+Z=180^\circ$, $\sigma_1-\beta_1=\sigma_2-\beta_2=\sigma_3-\beta_3$, $\sigma_1-\gamma_1=\sigma_2-\gamma_2=\sigma_3-\gamma_3$						
	$\begin{cases} a=13.2480, & x=6.0361 \\ b=11.5857, & y=11.4048 \\ c=9.7425, & z=9.5616 \end{cases}$	15° 45' 11.24"	16° 2' 3.76"	38° 27' 34.78"	38° 34' 4.24"	45° 48' 31.02"	35° 22' 33.80"	90.22281	A group of six tetra- hedra, all having one common pair of op- posite edges; and the other four edges having the same lengths in each but differently disposed.						
184.	$\begin{cases} a=13.2480, & x=9.5616 \\ b=11.5857, & y=11.4048 \\ c=9.7425, & z=9.5616 \end{cases}$	26° 12' 10.45"	28° 28' 42.82"	45° 56' 24.78"	38° 34' 4.24"	45° 48' 31.02"	35° 22' 33.80"	90.22281	The volumes are equal two and two; and in any pair of equal volume, the sum of two corre- sponding dihedral angles is supplement- ary. Several rela- tions of the form $A+X=B+Y$ hold.						
	$\begin{cases} a=13.2480, & x=6.0361 \\ b=11.5857, & y=11.4048 \\ c=9.5616, & z=9.7425 \end{cases}$	15° 43' 58.45"	16° 2' 7.71"	38° 7' 40.21"	38° 54' 50.80"	45° 48' 31.02"	35° 22' 33.80"	90.14519							
185.	$\begin{cases} a=13.2480, & x=9.5616 \\ b=11.5857, & y=11.4048 \\ c=9.5616, & z=9.7425 \end{cases}$	26° 55' 6.71"	29° 20' 3.57"	38° 34' 4.24"	45° 48' 31.02"	35° 22' 33.80"	35° 22' 33.80"	90.14519							
	$\begin{cases} a=13.2480, & x=6.0361 \\ b=11.5857, & y=11.4048 \\ c=6.0361, & z=9.7425 \end{cases}$	26° 55' 6.71"	29° 20' 3.57"	38° 34' 4.24"	45° 48' 31.02"	35° 22' 33.80"	35° 22' 33.80"	89.43556							

EXAMPLES. X.—Continued.

DATA.	$\frac{1}{2}a$, $\frac{1}{2}b$, $\frac{1}{2}c$	$\frac{1}{2}A$, $\frac{1}{2}B$, $\frac{1}{2}C$	$\frac{1}{2}a$, $\frac{1}{2}b$, $\frac{1}{2}c$	$\frac{1}{2}A$, $\frac{1}{2}B$, $\frac{1}{2}C$	$\frac{1}{2}a$, $\frac{1}{2}b$, $\frac{1}{2}c$	$\frac{1}{2}A$, $\frac{1}{2}B$, $\frac{1}{2}C$	VOLUME.	REMARKS.
211. $\begin{cases} a=7.70142, & x=5.618158 \\ b=8.5445, & y=6.6434 \\ c=7.6147, & z=5.7136 \end{cases}$	$20^{\circ} 5' 13.98''$ $25^{\circ} 41' 16.43''$ $20^{\circ} 22' 37.42''$	$26^{\circ} 43' 18.50''$ $38^{\circ} 44' 43.74''$ $31^{\circ} 47' 6.53''$	$30^{\circ} 52' 38.84''$ $39^{\circ} 53' 22.25''$ $30^{\circ} 29' 7.68''$	$32^{\circ} 31' 37.04''$ $39^{\circ} 53' 38.34''$ $27^{\circ} 23' 19.25''$	$33^{\circ} 36' 44.42''$ $46^{\circ} 56' 1.00''$ $33^{\circ} 6' 52.90''$	$27^{\circ} 35' 20.32''$ $41^{\circ} 54' 19.94''$ $28^{\circ} 5' 11.84''$	35.00578	$A - \lambda = C + Z$, $B - \lambda = C - Z$, $\sigma = .97158.71''$
212. $\begin{cases} a=8.9763, & x=5.06584 \\ b=8.9763, & y=4.7268 \\ c=8.01066, & z=5.09214 \end{cases}$	$17^{\circ} 3' 4.15''$ $15^{\circ} 14' 45.59''$ $18^{\circ} 6' 6.27''$	$28^{\circ} 38' 34.36''$ $34^{\circ} 19' 35.22''$ $29^{\circ} 39' 17.25''$	$37^{\circ} 24' 18.51''$ $25^{\circ} 51' 48.19''$ $31^{\circ} 17' 27.55''$	$45^{\circ} 5' 57.16''$ $41^{\circ} 43' 28.30''$ $35^{\circ} 29' 37.45''$	$46^{\circ} 14' 22.54''$ $40^{\circ} 30' 6.62''$ $33^{\circ} 12' 9.72''$	$31^{\circ} 51' 31.23''$ $27^{\circ} 7' 9.31''$ $35^{\circ} 25' 0.21''$	30	$A - \lambda = B - \lambda$, $B + \lambda = C + Z$
213. $\begin{cases} a=9.8033, & x=5.6138 \\ b=8.9763, & y=4.7268 \\ c=8.0107, & z=5.0924 \end{cases}$	$19^{\circ} 8' 23.15''$ $14^{\circ} 5' 38.70''$ $17^{\circ} 20' 32.92''$	$32^{\circ} 24' 40.59''$ $31^{\circ} 54' 21.84''$ $26^{\circ} 35' 27.42''$	$40^{\circ} 15' 5.24''$ $24^{\circ} 38' 31.14''$ $30^{\circ} 46' 45.45''$	$49^{\circ} 18' 53.88''$ $40^{\circ} 1' 41.45''$ $32^{\circ} 50' 48.27''$	$52^{\circ} 4' 27.50''$ $37^{\circ} 34' 48.01''$ $29^{\circ} 46' 34.41''$	$38^{\circ} 31' 35.02''$ $23^{\circ} 59' 55.59''$ $33^{\circ} 46' 9.25''$	31.06101	$A - \lambda = B - \lambda$, $B + \lambda = C + Z$
214. $\begin{cases} a=9.8671, & x=5.6176 \\ b=8.9763, & y=4.7268 \\ c=8.1807, & z=5.52218 \end{cases}$	$18^{\circ} 56' 0.90''$ $14^{\circ} 13' 32.89''$ $16^{\circ} 49' 52.50''$	$32^{\circ} 58' 16.63''$ $32^{\circ} 8' 36.50''$ $27^{\circ} 46' 14.72''$	$41^{\circ} 1' 31.00''$ $25^{\circ} 6' 8.84''$ $31^{\circ} 41' 1.90''$	$48^{\circ} 0' 12.39''$ $39^{\circ} 22' 57.20''$ $31^{\circ} 55' 34.53''$	$51^{\circ} 14' 52.86''$ $37^{\circ} 10' 15.09''$ $30^{\circ} 53' 36.10''$	$38^{\circ} 41' 44.17''$ $24^{\circ} 37' 6.40''$ $30^{\circ} 53' 45.39''$	31.12113	$A - \lambda = 180^{\circ}$, $C = Z$, so that this is the tetrahedron of maxi- mum volume under the conditions.
215. $\begin{cases} a=9.86936, & x=5.61986 \\ b=8.9763, & y=4.7268 \\ c=8.18092, & z=5.52218 \end{cases}$	$18^{\circ} 56' 29.39''$ $14^{\circ} 13' 16.98''$ $16^{\circ} 49' 38.57''$	$32^{\circ} 59' 15.00''$ $32^{\circ} 8' 1.60''$ $27^{\circ} 45' 35.96''$	$41^{\circ} 2' 19.83''$ $25^{\circ} 5' 50.30''$ $31^{\circ} 40' 57.77''$	$48^{\circ} 1' 7.06''$ $39^{\circ} 22' 32.81''$ $31^{\circ} 54' 54.70''$	$51^{\circ} 16' 21.93''$ $37^{\circ} 9' 7.50''$ $30^{\circ} 52' 42.75''$	$38^{\circ} 43' 31.69''$ $24^{\circ} 30' 17.66''$ $30^{\circ} 52' 42.81''$	31.12115	$a - x = b - y = 4.2495$, $c + z = 13.7031$.
216. $\begin{cases} a=9.869433, & x=5.619933 \\ b=8.9763, & y=4.7268 \\ c=8.180921, & z=5.522179 \end{cases}$	$18^{\circ} 56' 30.38''$ $14^{\circ} 13' 16.44''$ $16^{\circ} 49' 38.22''$	$32^{\circ} 59' 16.77''$ $32^{\circ} 8' 0.44''$ $27^{\circ} 45' 34.57''$	$41^{\circ} 2' 21.34''$ $25^{\circ} 5' 49.66''$ $31^{\circ} 40' 57.55''$	$48^{\circ} 1' 8.99''$ $39^{\circ} 22' 32.07''$ $31^{\circ} 54' 53.57''$	$51^{\circ} 16' 24.88''$ $37^{\circ} 9' 5.81''$ $30^{\circ} 52' 40.93''$	$38^{\circ} 43' 35.12''$ $24^{\circ} 36' 16.05''$ $30^{\circ} 52' 40.93''$	31.12115	$a - x = 4.2495$, $c + z = 13.7031$, $a - x = b - y = 4.2495$, $c + z = 13.7031$.
217. $\begin{cases} a=9.0481, & x=8.8901 \\ b=8.9217, & y=9.0797 \\ c=6.4025, & z=6.3040 \end{cases}$	$34^{\circ} 20' 5.56''$ $34^{\circ} 47' 38.38''$ $20^{\circ} 31' 59.27''$	$33^{\circ} 55' 43.65''$ $34^{\circ} 11' 43.75''$ $20^{\circ} 40' 55.67''$	$35^{\circ} 16' 16.08''$ $35^{\circ} 32' 34.11''$ $21^{\circ} 3' 58.83''$	$34^{\circ} 31' 35.95''$ $34^{\circ} 55' 55.55''$ $20^{\circ} 31' 42.24''$	$40^{\circ} 47' 59.34''$ $40^{\circ} 59' 25.55''$ $22^{\circ} 22' 6.00''$	$40^{\circ} 17' 31.62''$ $41^{\circ} 29' 53.22''$ $22^{\circ} 2' 35.08''$	52.31824	$I = I' = 180^{\circ}$, $A_1 + B_1 = 180^{\circ}$, $B_1 + C_1 = 180^{\circ}$, $B_2 - A_2 = 0$
218. $\begin{cases} a=9.0481, & x=8.8901 \\ b=8.9217, & y=9.0797 \\ c=6.4025, & z=6.3040 \end{cases}$	$34^{\circ} 20' 5.56''$ $22^{\circ} 4' 46.20''$ $30^{\circ} 20' 56.00''$	$33^{\circ} 55' 43.65''$ $33^{\circ} 28' 55.97''$ $22^{\circ} 31' 2.78''$	$35^{\circ} 16' 16.08''$ $35^{\circ} 32' 34.11''$ $21^{\circ} 3' 58.89''$	$45^{\circ} 24' 11.02''$ $34^{\circ} 55' 55.55''$ $35^{\circ} 32' 34.11''$	$40^{\circ} 12' 0.66''$ $44^{\circ} 24' 21.68''$ $24^{\circ} 53' 46.04''$	$44^{\circ} 24' 21.68''$ $44^{\circ} 24' 21.68''$ $34^{\circ} 41' 27.14''$	52.31824	
219. $\begin{cases} a=9.0481, & x=6.3040 \\ b=8.9217, & y=9.0797 \\ c=6.4025, & z=8.8901 \end{cases}$	$22^{\circ} 28' 34.72''$ $31^{\circ} 47' 28.38''$ $29^{\circ} 38' 56.74''$	$20^{\circ} 31' 42.24''$ $29^{\circ} 49' 17.24''$ $20^{\circ} 40' 55.67''$	$30^{\circ} 31' 46.02''$ $35^{\circ} 32' 34.11''$ $22^{\circ} 3' 58.83''$	$34^{\circ} 31' 35.95''$ $34^{\circ} 55' 55.55''$ $33^{\circ} 55' 43.65''$	$33^{\circ} 49' 28.48''$ $49^{\circ} 0' 34.44''$ $24^{\circ} 39' 35.45''$	$24^{\circ} 27' 25.49''$ $45^{\circ} 33' 16.57''$ $33^{\circ} 15' 57.76''$	52.31824	
220. $\begin{cases} a=9.0481, & x=8.8901 \\ b=8.9217, & y=9.0797 \\ c=6.3040, & z=6.4025 \end{cases}$	$34^{\circ} 30' 7.54''$ $34^{\circ} 57' 35.22''$ $20^{\circ} 52' 8.21''$	$33^{\circ} 46' 30.08''$ $34^{\circ} 2' 13.30''$ $20^{\circ} 20' 58.97''$	$35^{\circ} 5' 38.49''$ $35^{\circ} 32' 38.40''$ $20^{\circ} 43' 37.91''$	$34^{\circ} 31' 25.81''$ $34^{\circ} 46' 14.55''$ $20^{\circ} 51' 51.46''$	$40^{\circ} 47' 27.33''$ $40^{\circ} 59' 57.18''$ $22^{\circ} 3' 9.31''$	$40^{\circ} 18' 3.28''$ $41^{\circ} 29' 21.25''$ $22^{\circ} 21' 14.65''$	52.31891	See next page.

EXAMPLES. X.—Continued.

DATA.	$\frac{1}{2}a_1, \frac{1}{2}b_1, \frac{1}{2}c_1$	$\frac{1}{2}a_2, \frac{1}{2}b_2, \frac{1}{2}c_2$	$\frac{1}{2}a_3, \frac{1}{2}b_3, \frac{1}{2}c_3$	$\frac{1}{2}a_4, \frac{1}{2}b_4, \frac{1}{2}c_4$	$\frac{1}{2}A, \frac{1}{2}B, \frac{1}{2}C$	$\frac{1}{2}X, \frac{1}{2}Y, \frac{1}{2}Z$	VOLUME.	REMARKS.
231. $\begin{cases} a=3.6683, \\ b=3.4069, \\ c=2.5535, \end{cases}$	$\begin{cases} x=4.1154, \\ y=4.4058, \\ z=5.2602 \end{cases}$	$\begin{cases} 43^{\circ}26'47.72'' \\ 41^{\circ}53'76'' \\ 47^{\circ}59'33.52'' \end{cases}$	$\begin{cases} 24^{\circ}54'55.35'' \\ 20^{\circ}3'2.79'' \\ 17^{\circ}41'48.35'' \end{cases}$	$\begin{cases} 21^{\circ}57'23.69'' \\ 27^{\circ}34'28.59'' \\ 37^{\circ}55'21.53'' \end{cases}$	$\begin{cases} 29^{\circ}17'2.61'' \\ 28^{\circ}39'9.86'' \\ 23^{\circ}53'13.44'' \end{cases}$	$\begin{cases} 43^{\circ}20'52.46'' \\ 43^{\circ}58'45.21'' \\ 48^{\circ}2'41.63'' \end{cases}$	5 278617	$A + X = B - Y = C + Z$ $\sigma_1 - \sigma_2 = 12'55.70''$
232. $\begin{cases} a=3.6683, \\ b=3.4069, \\ c=3.5485, \end{cases}$	$\begin{cases} x=4.1454 \\ y=4.4068 \\ z=5.2632 \end{cases}$	$\begin{cases} 43^{\circ}29'8.44'' \\ 44^{\circ}11'8.50'' \\ 48^{\circ}3'11.86'' \end{cases}$	$\begin{cases} 24^{\circ}53'44.94'' \\ 20^{\circ}1'20.91'' \\ 17^{\circ}39'27.76'' \end{cases}$	$\begin{cases} 21^{\circ}55'27.23'' \\ 27^{\circ}34'39.37'' \\ 37^{\circ}55'12.18'' \end{cases}$	$\begin{cases} 29^{\circ}15'29.79'' \\ 28^{\circ}37'47.10'' \\ 24^{\circ}32'42.89'' \end{cases}$	$\begin{cases} 43^{\circ}23'50.90'' \\ 44^{\circ}1'3.59'' \\ 48^{\circ}6'7.80'' \end{cases}$	5 267735	$\sigma_1 - \sigma_2 = 12'56.51''$
233. $\begin{cases} a=3.6683, \\ b=3.4069, \\ c=3.5481, \end{cases}$	$\begin{cases} x=4.1454 \\ y=4.4068 \\ z=5.2636 \end{cases}$	$\begin{cases} 43^{\circ}29'19.73'' \\ 44^{\circ}11'19.30'' \\ 48^{\circ}3'29.36'' \end{cases}$	$\begin{cases} 24^{\circ}53'39.30'' \\ 20^{\circ}1'12.74'' \\ 17^{\circ}39'16.50'' \end{cases}$	$\begin{cases} 21^{\circ}55'17.90'' \\ 27^{\circ}34'40.23'' \\ 37^{\circ}58'6.04'' \end{cases}$	$\begin{cases} 29^{\circ}15'22.31'' \\ 28^{\circ}37'40.46'' \\ 24^{\circ}32'30.81'' \end{cases}$	$\begin{cases} 43^{\circ}23'32.83'' \\ 44^{\circ}1'14.68'' \\ 48^{\circ}6'24.33'' \end{cases}$	5 266861	$\sigma_1 - \sigma_2 = 12'56.55''$ being the max. value when a, b, x, y , and $c + z$ have the values here assigned them.
234. $\begin{cases} a=3.6683, \\ b=3.4069, \\ c=2.0007, \end{cases}$	$\begin{cases} x=4.1454 \\ y=4.4068 \\ z=5.8130 \end{cases}$	$\begin{cases} 48^{\circ}15'58.69'' \\ 48^{\circ}30'17.22'' \\ 55^{\circ}13'6.02'' \end{cases}$	$\begin{cases} 22^{\circ}39'30.65'' \\ 16^{\circ}39'21.14'' \\ 13^{\circ}22'11.05'' \end{cases}$	$\begin{cases} 18^{\circ}7'32.84'' \\ 24^{\circ}32'58.38'' \\ 14^{\circ}19'23.34'' \end{cases}$	$\begin{cases} 27^{\circ}47'31.73'' \\ 27^{\circ}21'37.97'' \\ 42^{\circ}47'30.97'' \end{cases}$	$\begin{cases} 50^{\circ}1'39.56'' \\ 50^{\circ}23'2.84'' \\ 55^{\circ}52'9.85'' \end{cases}$	3 810566	$\sigma_1 - \sigma_2 = 3^{\circ}54'30.32''$
235. $\begin{cases} a=3.3457, \\ b=2.94202, \\ c=3.4569, \end{cases}$	$\begin{cases} x=4.4568 \\ y=4.85958 \\ z=4.3456 \end{cases}$	$\begin{cases} 43^{\circ}36'48.39'' \\ 43^{\circ}35'3.08'' \\ 43^{\circ}35'16.57'' \end{cases}$	$\begin{cases} 28^{\circ}47'43.41'' \\ 21^{\circ}16'52.73'' \\ 22^{\circ}40'0.20'' \end{cases}$	$\begin{cases} 25^{\circ}7'50.70'' \\ 31^{\circ}30'9.13'' \\ 25^{\circ}24'28.98'' \end{cases}$	$\begin{cases} 21^{\circ}44'56.80'' \\ 26^{\circ}35'39.85'' \\ 28^{\circ}39'59.43'' \end{cases}$	$\begin{cases} 28^{\circ}18'54.85'' \\ 45^{\circ}38'14.18'' \\ 43^{\circ}33'54.60'' \end{cases}$	5 660726	$\sigma_1 = \sigma_2$
236. $\begin{cases} a=3.72374, \\ b=5.19556, \\ c=4.87204, \end{cases}$	$\begin{cases} x=3.672916 \\ y=4.87204 \\ z=3.44460 \end{cases}$	$\begin{cases} 22^{\circ}6'11.34'' \\ 39^{\circ}0'26.71'' \\ 20^{\circ}44'0.26'' \end{cases}$	$\begin{cases} 24^{\circ}23'47.02'' \\ 46^{\circ}24'36.89'' \\ 29^{\circ}20'57.60'' \end{cases}$	$\begin{cases} 22^{\circ}51'22.87'' \\ 43^{\circ}9'57.53'' \\ 27^{\circ}38'42.41'' \end{cases}$	$\begin{cases} 23^{\circ}38'35.69'' \\ 40^{\circ}15'8.05'' \\ 22^{\circ}26'15.45'' \end{cases}$	$\begin{cases} 18^{\circ}21'7.22'' \\ 62^{\circ}16'7.63'' \\ 17^{\circ}17'6.82'' \end{cases}$	5 310628	
237. $\begin{cases} a=3.72374, \\ b=5.19556, \\ c=4.87204, \end{cases}$	$\begin{cases} x=3.55213 \\ y=4.33034 \\ z=3.67124 \end{cases}$	$\begin{cases} 20^{\circ}34'54.15'' \\ 29^{\circ}20'57.60'' \\ 22^{\circ}30'2.36'' \end{cases}$	$\begin{cases} 25^{\circ}57'2.89'' \\ 44^{\circ}36'36.29'' \\ 37^{\circ}0'20.71'' \end{cases}$	$\begin{cases} 22^{\circ}53'21.35'' \\ 36^{\circ}48'16.75'' \\ 32^{\circ}15'48.71'' \end{cases}$	$\begin{cases} 23^{\circ}38'35.69'' \\ 37^{\circ}9'17.14'' \\ 27^{\circ}14'40.36'' \end{cases}$	$\begin{cases} 25^{\circ}10'43.95'' \\ 45^{\circ}47'13.01'' \\ 27^{\circ}54'34.32'' \end{cases}$	7 30914	
238. $\begin{cases} a=7.3782, \\ b=9.2160, \\ c=8.1664, \end{cases}$	$\begin{cases} x=6.5108 \\ y=5.4688 \\ z=5.7472 \end{cases}$	$\begin{cases} 21^{\circ}50'40.74'' \\ 20^{\circ}24'3.68'' \\ 19^{\circ}16'47.88'' \end{cases}$	$\begin{cases} 35^{\circ}38'56.37'' \\ 44^{\circ}8'38.64'' \\ 38^{\circ}40'52.69'' \end{cases}$	$\begin{cases} 26^{\circ}34'33.48'' \\ 26^{\circ}11'4.04'' \\ 29^{\circ}47'40.98'' \end{cases}$	$\begin{cases} 30^{\circ}55'3.63'' \\ 38^{\circ}21'38.28'' \\ 28^{\circ}9'59.59'' \end{cases}$	$\begin{cases} 32^{\circ}24'30.24'' \\ 46^{\circ}14'55.80'' \\ 35^{\circ}37'57.94'' \end{cases}$	35 68824	See next page.
239. $\begin{cases} a=7.3543, \\ b=8.0000, \\ c=4.4722, \end{cases}$	$\begin{cases} x=7.73156 \\ y=6.9282 \\ z=6.0000 \end{cases}$	$\begin{cases} 35^{\circ}6'22.88'' \\ 32^{\circ}15'35.19'' \\ 22^{\circ}38'1.11'' \end{cases}$	$\begin{cases} 36^{\circ}31'49.43'' \\ 35^{\circ}38'55.72'' \\ 17^{\circ}49'15.70'' \end{cases}$	$\begin{cases} 31^{\circ}43'3.17'' \\ 39^{\circ}55'9.11'' \\ 36^{\circ}24'21.15'' \end{cases}$	$\begin{cases} 39^{\circ}55'9.11'' \\ 28^{\circ}9'59.59'' \\ 35^{\circ}37'57.94'' \end{cases}$	$\begin{cases} 44^{\circ}3'56.80'' \\ 46^{\circ}40'23.11'' \\ 15^{\circ}43'47.09'' \end{cases}$	28.84288	
240. $\begin{cases} a=4.2893, \\ b=5.0794, \\ c=5.6318, \end{cases}$	$\begin{cases} x=3.931876 \\ y=3.9794 \\ z=4.1911 \end{cases}$	$\begin{cases} 20^{\circ}20'23.19'' \\ 22^{\circ}24'5.55'' \\ 23^{\circ}37'55.51'' \end{cases}$	$\begin{cases} 28^{\circ}43'4.96'' \\ 42^{\circ}1'20.41'' \\ 42^{\circ}53'10.41'' \end{cases}$	$\begin{cases} 24^{\circ}20'44.08'' \\ 29^{\circ}16'6.80'' \\ 34^{\circ}30'17.68'' \end{cases}$	$\begin{cases} 24^{\circ}42'44.07'' \\ 35^{\circ}9'19.13'' \\ 32^{\circ}0'48.24'' \end{cases}$	$\begin{cases} 28^{\circ}49'49.97'' \\ 32^{\circ}59'22.04'' \\ 36^{\circ}5'31.88'' \end{cases}$	9 999143	

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